Curved Koszul duality theory

Joseph Hirsh, Joan Millès

CURVED KOSZUL DUALITY THEORY

JOSEPH HIRSH AND JOAN MILLÈS

Abstract. We extend the bar-cobar adjunction to operads and properads, not necessarily augmented. Due to the default of augmentation, the objects of the dual category are endowed with a curvature. As usual, the bar-cobar construction gives a cofibrant resolution for any properad. Applied to the properad encoding unital and counital Frobenius algebras, notion which appears in 2d-TQFT, it defines the associated notion up to homotopy. We further define a curved Koszul duality theory for operads or properads presented with quadratic, linear and constant relations. This provides smaller resolutions. We apply this new theory to study the homotopy theory and the cohomology theory of unital associative algebras.

MSC: 18D50, 18G10

Introduction

In [Hoc45], Hochschild introduced a (co)homology theory for associative algebras and in [Sta63], Stasheff introduced the homotopy theory for associative algebras. Nowadays, we know how to describe these theories in operadic terms, but this approach does not encode the units in unital associative algebras. In order to define a homotopy theory and a cohomology theory for unital associative algebras, we refine the operadic theory and more precisely its Koszul duality theory.

In representation theory, an algebra A is "represented" as an algebra of operations with one input and one output on a vector space V via a representation $\mu \in \mathrm{Hom}_{\mathsf{Alg.}}(A, \mathrm{End}(V))$. To encode operations with several inputs and one output, one uses the notion of an operad [May72, BV73]. More generally, one uses the notion of properads to encode operations with several inputs and several outputs [Val07]. An associative algebra is a special kind of operad and an operad is a special kind of properad, and theories about properads generalize those of operads and associative algebras. For example a bar-cobar adjunction defined in a properadic setting generalizes one defined for operads and algebras. In [Val07], the bar construction B assigned a coaugmented dg coproperad to an augmented dg properad and the cobar construction Ω assigns an augmented dg properad to a coaugmented dg coproperad, and the two constructions are adjoint. An important property of the adjunction is that the bar-cobar composition $\Omega \mathrm{B}\mathcal{P}$ defines a cofibrant resolution of an augmented dg properad $\mathcal{P}$.

In this paper, we extend the bar-cobar adjunction (Ω, B) to non-augmented properads. The lack of augmentation appears on the new bar construction as a *curvature*. We therefore define *curved coproperad*, whose our bar construction is an example. We then extend the cobar construction to coaugmented curved coproperads, resulting in a dg properad. The composition bar-cobar provides a cofibrant resolution $\Omega \mathrm{B}\mathcal{P}$ of a properad $\mathcal{P}$. For example, we obtain a cofibrant resolution for the properad encoding unital and/or counital Frobenius algebras. Since the datum of a 2-dimensional topological quantum field theory, 2d-TQFT for short, is equivalent to a unital and counital Frobenius algebra structure [Abr96, Koc04], this provides homotopy tools to study 2d-TQFT. With our model, the methods of [Wil07] apply to show that the differential forms $\Omega_{dR}(M)$ on a closed, oriented manifold M bear a unital and counital Frobenius algebra structure up to homotopy.

The first author was supported by a National Science Foundation Graduate Research Fellowship.
The second is supported by the ANR grant JCJC06 OBTH.

The bar-cobar resolution $\Omega \mathrm{B} \mathcal{P}$ is large and it is often desirable to have a smaller resolution. To this end, we develop a curved Koszul duality theory for properads generalizing the Koszul duality theory for properads [Val07], operads [GJ94, GK94], and associative algebras [Pri70]. One of the main object is the Koszul dual coproperad $\mathcal{P}^{\text{!`}}$, which has, here, a curvature. It applies to properads with a quadratic, linear and constant presentation. The properads for which this theory apply are called Koszul properads. In this case, the cobar construction $\Omega \mathcal{P}^{\text{!`}}$ is a resolution of $\mathcal{P}$. This theory extends to coloured operads to recover, as an example, the resolution of the coloured operad $\mathcal{I}so$ given by Markl in [Mar01]. We summarize the different generalizations of the Koszul duality theory in the following table:

<table>
<tr><th>Relations / Monoids</th><th>Homogeneous quadratic</th><th>Quadratic and linear</th><th>Quadratic, linear and constant</th></tr>
<tr><td>Associative algebras</td><td colspan="2">[Pri70]</td><td>[Pos93, PP05]</td></tr>
<tr><td>Operads</td><td>[GJ94, GK94]</td><td rowspan="2">[GCTV09]</td><td rowspan="2">Section 4 of this paper</td></tr>
<tr><td>Properads</td><td>[Val07]</td></tr>
</table>

The operad $u\mathcal{A}s$ encoding unital associative algebras is an example of an operad with quadratic, linear and constant relations. It is a Koszul operad in the previous sense and we get a "small" cofibrant resolution $uA_\infty := \Omega u\mathcal{A}s^{\text{!`}} \xrightarrow{\sim} u\mathcal{A}s$. This particularly simple resolution allows us to define the notion of *homotopy unital associative algebras*. We recover actually the notion of homotopy unit for A_∞-algebra which appears in [FOOO09]. After we achieved this work, we were told about the existence of the incoming paper of Lyubashenko [Lyu10]. In this paper, the author extracts, from the definition of [FOOO09], an operad, which corresponds to the one given by the present Koszul duality theory, and proves that it is a cofibrant resolution of the operad $u\mathcal{A}s$. However, with our approach in terms of Koszul duality theory, we prove several of the good homotopy properties that carry algebras over a cofibrant operad (rectification, transfer, "strict" minimal model). We also obtain functorial resolutions on the level of unital associative algebras. We use these other resolutions to study the cohomology theory of unital associative algebras.

We begin the paper with a survey of the results on homotopy unital associative algebras expressed in an internal language, explained without, for example, the words "operad" or "properad". This section corresponds to the results obtained in the last section of this paper. In Section 2, we recall definitions of associative algebras, operads and properads. In Section 3, we extend the bar and the cobar construction to the non-augmented framework and we define the notion of curved twisting morphims. In Section 4, we extend the Koszul duality theory for homogeneous quadratic properads to properads with quadratic, linear and constant relations. Section 5 is devoted to resolution of non-augmented properads as bimodules over themselves and to functorial resolutions of $\mathcal{P}$-algebras. Section 6 studies the operad encoding unital associative algebras. We describe the homotopy theory and the cohomology theory for this category of algebras.

In this paper, we work over a field $\mathbb{K}$ of characteristic 0.

Contents

1. Results on unital associative algebras

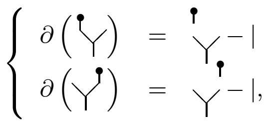
In this section, we develop the homotopy and cohomology theories of unital associative algebras. The definitions, proofs, techniques, and pictorial descriptions of the results are based on operad theory and can be found in Section 6. However, this section does not contain the word "operad" and can be read independently from the rest of the paper. The comparison with the work of [FOOO09] is described in Section 6.

1.1. **Unital associative algebra.** A *unital associative differential graded algebra*, or *unital dga*, is a quadruple $(A,\, \mu,\, e,\, d_A)$, where $(A,\, d_A)$ is a dg module, $\mu : A \otimes A \to A$, and $e : \mathbb{K} \to A$ are dg module maps, such that the map μ is associative and such that the element $e(1_{\mathbb{K}})$ is a left and right unit for the associative product μ.

The version of this structure "up to homotopy" is what we call a *uA_∞-algebra*, for *homotopy unital associative algebra*. Let $f : V \to W$ be a homogeneous $\mathbb{K}$-linear map of degree $|f|$. We denote its derivative by $\partial(f) := d_W \circ f - (-1)^{|f|} f \circ d_V$.

1.2. **Homotopy unital associative algebra.** A *homotopy unital associative algebra* or *uA_∞-algebra structure on a dg module* $(A,\, d_A)$ is given by a collection of maps $\{\mu_n^S : A^{\otimes(n-|S|)} \to A\}$ of degree $n-2+|S|$, where the set S runs over the set of subsets of $\{1,\, \ldots,\, n\}$ for any integer $n \geq 2$ and where $S = \{1\}$ when $n = 1$. The μ_n^S are given pictorially by planar corollas with n entries labelled by $1, \cdots, n$ on which we put "corks" when the label is in S. For example, we have $\mu_3^{\{1\}} =$ . The maps μ_n^S satisfy the following identities:

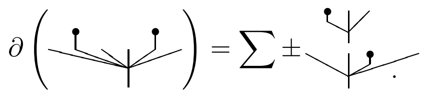
- $\mu_2^{\{1\}}$ and $\mu_2^{\{2\}}$ are homotopies for the unit



  where the empty space between the corollas and the corks is the composition of operations and where $|$ is the identity of A
- for $(n,\, S) \neq (2,\, \{1\})$ and $(n,\, S) \neq (2,\, \{2\})$,

$$\partial(\mu_n^S) = \sum_{\substack{p+q+r=n\\ p+1+r=m}} (-1)^{q(r+|S_1|)+|S_2||S_1'|+p+1} \mu_m^{S_1} \circ (\underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p-|S_1'|},\, \mu_q^{S_2},\, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{r-|S_1''|}).$$

Or, pictorially:



Examples.

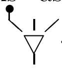
(1) Every unital dga $(A,\, \mu,\, e, d_A)$ naturally equips the dg module (A, d_A) with the structure of a uA_∞-algebra by

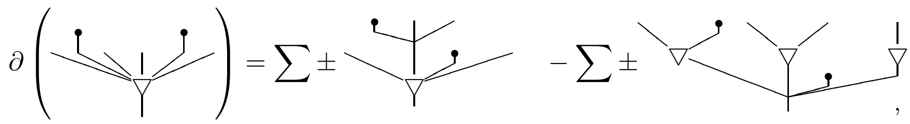
$$\mu_n^S = \begin{cases} \mu & \text{if } n = 2 \text{ and } S = \emptyset \\ e & \text{if } n = 1 \text{ and } S = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

(2) A *strictly unital A_∞-algebra*, or $\mathtt{su}A_\infty$-algebra is an A_∞-algebra $(A, d_A, \{\mu_n\}_{n\geq 2})$ with $e \in A$ so that e is a left and right unit for μ_2, and e annihilates μ_n for $n \geq 3$ [KS06]. Every $\mathtt{su}A_\infty$-algebra is naturally a uA_∞-algebra by

$$\mu_n^S = \begin{cases} \mu_n & \text{if } n \geq 2, S = \emptyset \\ e & \text{if } n = 1 \text{ and } S = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

Remark. Every uA_∞-algebra contains an A_∞-algebra if we take $\mu_n := \mu_n^\emptyset$ for all $n \geq 1$. The additional algebraic structure given by a uA_∞-algebra provides homotopies for the "unital relations" along with the homotopies already present for the "associative relations."

1.3. **Infinity-morphism.** We define the notion of infinity-morphism between two uA_∞-algebras A and B by a collection of maps $f_n^S : A^{\otimes(n-|S|)} \to B$ of degree $n-1+|S|$, represented graphically by planar trees with "corks" as the uA_∞-algebra structures but with a triangle $\bigtriangledown$ as vertex. For example, we have $f_3^{\{1\}} =$ . The f_n^S satisfy the relations:



where the planar trees with "corks" and no triangle represent the uA_∞-algebra structure of A on the top and the uA_∞-algebra structure of B on the bottom. With this definition of infinity-morphism, we prove a rectification theorem.

1.3.1. **Theorem** (Rectification Theorem, Theorem 6.3.2). *Let A be a uA_∞-algebra. We can rectify A: there is a unital associative algebra A' such that A is uA_∞-equivalent to A'.*

Moreover, we have a transfer theorem.

1.3.2. **Theorem** (Homotopy Transfer Theorem, Theorem 6.4.5). *Let A be a homotopy unital associative algebra and let V be a chain complex. Given a strong deformation retract*

$$\begin{tikzcd} V \arrow[r, shift left, "i"] & A \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd},$$

i.e., p and i are chain maps, where $p \circ i = \mathrm{id}_V$ and $d_A h + h d_A = \mathrm{id}_A - i \circ p$, there is a natural uA_∞-algebra structure on V, and a natural extension of i to an infinity-morphism.

1.4. **Comparison with the literature.** In the literature, there are several definitions of "weakly unital" or "homotopy unital" A_∞-algebras [KS06, Lyu02, Fuk02, FOOO09]. The definitions of [KS06] and [Lyu02], describe *properties* of A_∞-algebras, while the definition presented in [FOOO09] describes a *structure* on an A_∞-algebra. In [LM06] these are compared and shown to be, in some sense, equivalent. Our notion of homotopy unital associative algebra, or uA_∞-algebra, is an A_∞-algebra with additional structure, and in fact coincides with the structure described in [FOOO09].

In [FOOO09], the authors prove (Theorem 5.4.2') that there is a (gapped, filtered) $\mathtt{su}A_\infty$ minimal model for every (gapped, filtered) uA_∞-algebra. We prove the following analogue.

1.4.1. **Theorem** (Corollary 6.5.3). *Let A be a uA_∞-algebra. There is an $\mathtt{su}A_\infty$-algebra structure on the homology of A which is equivalent to A.*

We extend this theorem to a broad class of algebraic structures, including Batalin-Vilkovisky algebras and commutative algebras.

1.5. **André-Quillen cohomology theory for unital associative algebra.** Following the ideas of Quillen, we define a cohomology theory associated to any unital associative dga A with coefficients in a A-bimodule M, denoted $\mathrm{H}^\bullet_{u\mathcal{A}s}(A, M)$. We prove that this cohomology theory is an Ext-functor and that it is equal to the Hochschild cohomology theory of the associative algebra A.

1.5.1. **Theorem** (Theorem 6.6.7). *Let A be a unital associative dga. We have*

$$\mathrm{H}^\bullet_{u\mathcal{A}s}(A, M) \cong \mathrm{HH}^{\bullet+1}(A, M).$$

2. Operads and Properads

In this section, we recall the notion of algebra, operad and properad as successive generalizations. We refer to the book of Loday and Vallette [LV] for a complete and modern exposition about algebras and operads in $\mathsf{dg\ mod}$, to the book of [MSS02] for another presentation and to the thesis of Vallette [Val07] for properads.

2.1. **Algebras.** Let $\mathbb{K}$-mod denote the monoidal category ($\mathbb{K}$-mod, $\otimes_{\mathbb{K}}$, $\mathbb{K}$) of $\mathbb{K}$-modules. A *unital associative algebra* is a monoid (A, μ, e) in this monoidal category. The product $\mu : A \otimes_{\mathbb{K}} A \to A$ is associative and $e : \mathbb{K} \to A$ is a *unit* for the product.

As in representation theory, the elements of A are seen as operations with one input and one output. Then we represent the product $a_1 \cdots a_n$ by a vertical bivalent tree whose vertices are indexed by the a_i, see Figure 1.

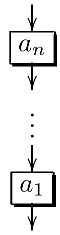


FIGURE 1. Representation of the product $a_1 \cdots a_n$

2.2. **Operads.** An $\mathbb{S}$-*module* $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ is a collection of $\mathbb{K}$-modules $\mathcal{P}(n)$ endowed with right action of the symmetric group $\mathbb{S}_n$. One defines from [May72] the monoidal product $\circ$ on the category of $\mathbb{S}$-modules by

$$(\mathcal{P} \circ \mathcal{Q})(n) := \bigoplus_{k \geq 0} \left(\mathcal{P}(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1 + \cdots + i_k = n} (\mathcal{Q}(i_1) \otimes \cdots \otimes \mathcal{Q}(i_k)) \otimes_{\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}} \mathbb{K}[\mathbb{S}_n] \right) \right),$$

where the notation $\otimes_{\mathbb{S}_k}$ stands for the space of coinvariants under the (diagonal) action of the symmetric group $\mathbb{S}_k$:

$$(p \otimes q_1 \otimes \cdots \otimes q_k \otimes \sigma) \cdot \nu := p \cdot \nu \otimes q_{\nu^{-1}(1)} \otimes \cdots \otimes q_{\nu^{-1}(k)} \otimes \bar{\nu}^{-1} \cdot \sigma$$

for any $p \in \mathcal{P}(k)$, $q_j \in \mathcal{Q}(i_j)$, $\sigma \in \mathbb{S}_n$ and $\nu \in \mathbb{S}_k$ with $\bar{\nu} \in \mathbb{S}_n$ the induced block-wise permutation. This monoidal product encodes the composition of multilinear operations and we represent it by 2-levels trees as shown in Figure 2.

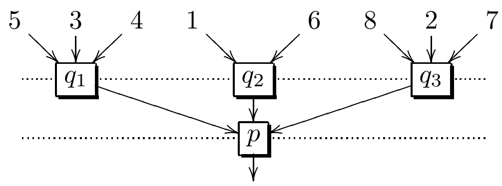


FIGURE 2. An element in $(\mathcal{P} \circ \mathcal{Q})(8)$

The unit for the monoidal product is $I := (0, \mathbb{K}, 0, \ldots)$ where the $\mathbb{K}$ is in arity 1 and represent the identity element modeled by the tree $|$. It forms a monoidal category denoted by $\mathbb{S}$-Mod.

An *operad* is a monoid $(\mathcal{P}, \gamma, e)$ in the monoidal category of $\mathbb{S}$-modules $\mathbb{S}$-Mod. The associative product $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ is called the *composition product* and $e : I \to \mathcal{P}$ is the *unit* for the composition product.

EXAMPLE. A unital associative algebra induces an operad by this injective map

$$\mathsf{Unital\ associative\ algebras} \rightarrowtail \mathsf{Operads}, \quad A \mapsto (0, A, 0, \ldots)$$

2.3. **Properads.** Algebras encode operations with one input and one output. Operads encode operations with several inputs and one output. To encode operations with multiple inputs and outputs, one uses the notion of *properad*.

An $\mathbb{S}$-*bimodule* $\mathcal{P}$ is a collection $\{\mathcal{P}(m,\, n)\}_{m,n\geq 0}$ of $\mathbb{S}_m$-$\mathbb{S}_n$-bimodules. One recalls from [Val07] a monoidal product using 2-levels graphs as in Figure 3.

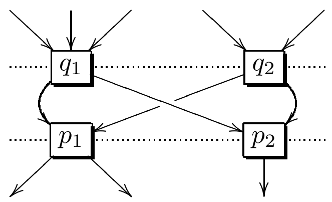


FIGURE 3. An element in $(\mathcal{P} \boxtimes \mathcal{Q})(3,\, 5)$

Let a and b the number of vertices on the first level and on the second level respectively. Let N be the number of internal edges between the two levels. We associate to an a-tuple of integers $\bar\imath = (i_1,\, \ldots,\, i_a)$ the sum $|\bar\imath| := i_1 + \cdots + i_a$. To any pair of a-tuples $\bar\imath$ and $\bar\jmath$ we denote by $\mathcal{P}(\bar\jmath,\, \bar\imath)$ the tensor product $\mathcal{P}(j_1,\, i_1) \otimes \cdots \otimes \mathcal{P}(j_a,\, i_a)$ and by $\mathbb{S}_{\bar\imath}$ the image of $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_a}$ in $\mathbb{S}_{|\bar\imath|}$.

Let $\bar k = (k_1,\, \ldots,\, k_b)$ be a b-tuple and let $\bar\jmath = (j_1,\, \ldots,\, j_a)$ be an a-tuple such that $|\bar k| = |\bar\jmath| = N$. A $(\bar k,\, \bar\jmath)$-*connected permutation* is a permutation σ in $\mathbb{S}_N$ such that the graph of a geometric representation of σ is connected when one connects the inputs labelled by $j_1+\cdots+j_i+1,\, \ldots,\, j_1+\cdots+j_{i+1}$ for $0 \leq i \leq a-1$ and the outputs labelled by $k_1+\cdots+k_i+1,\, \ldots,\, k_1+\cdots+k_{i+1}$ for $0 \leq i \leq b-1$. We denote by $\mathbb{S}^c_{\bar k,\bar\jmath}$ the set of $(\bar k,\, \bar\jmath)$-connected permutations.

We define the monoidal product $\boxtimes$, denoted $\boxtimes_c$ in [Val07], on the category of $\mathbb{S}$-bimodules by

$$(\mathcal{P} \boxtimes \mathcal{Q})(m,\, n) := \bigoplus_{N\in\mathbb{N}} \left(\bigoplus_{\bar l, \bar k, \bar\jmath, \bar\imath} \mathbb{K}[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar l}} \mathcal{P}(\bar l,\, \bar k) \otimes_{\mathbb{S}_{\bar k}} \mathbb{K}[\mathbb{S}^c_{\bar k,\bar\jmath}] \otimes_{\mathbb{S}_{\bar\jmath}} \mathcal{Q}(\bar\jmath,\, \bar\imath) \otimes_{\mathbb{S}_{\bar\imath}} \mathbb{K}[\mathbb{S}_n] \right)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a},$$

where the second direct sum runs over the b-tuples $\bar l$, $\bar k$ and the a-tuples $\bar\jmath$, $\bar\imath$ such that $|\bar l| = m$, $|\bar k| = |\bar\jmath| = N$, $|\bar\imath| = n$ and we consider the module of coinvariants with respect to the $\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a$-action:

$$\rho\otimes p_1\otimes\cdots\otimes p_b\otimes\sigma\otimes q_1\otimes\cdots\otimes q_a\otimes\omega \sim \rho\cdot\tau_{\bar l}^{-1}\otimes p_{\tau(1)}\otimes\cdots\otimes p_{\tau(b)}\otimes\tau_{\bar k}\cdot\sigma\cdot\nu_{\bar\jmath}\otimes q_{\nu^{-1}(1)}\otimes\cdots\otimes q_{\nu^{-1}(a)}\otimes\nu_{\bar\imath}^{-1}\cdot\omega,$$

for $\rho \in \mathbb{S}_m$, $\omega \in \mathbb{S}_n$, $\sigma \in \mathbb{S}^c_{\bar k,\bar\jmath}$ and for $\tau \in \mathbb{S}_b$ with $\tau_{\bar k}$ the associated block-wise permutation, $\nu \in \mathbb{S}_a$ with $\nu_{\bar\jmath}$ the associated block-wise permutation. We write an element in $\mathcal{P} \boxtimes \mathcal{Q}$ like this $\theta(p_1,\, \ldots,\, p_b)\sigma(q_1,\, \ldots,\, q_a)\omega$. The unit I for the monoidal product is given by

$$\begin{cases} I(1,\,1) & := \quad \mathbb{K} \quad \text{and} \\ I(m,\,n) & := \quad 0 \quad \text{otherwise.} \end{cases}$$

The category of $\mathbb{S} -$ *bimodules* with the operation $\boxtimes$ forms a monoidal category with unit I. We denote this monoidal category by $\mathbb{S}$-biMod.

A *properad* is a monoid $(\mathcal{P},\, \gamma,\, e)$ in the monoidal category $\mathbb{S}$-biMod of $\mathbb{S}$-bimodules. The associative product $\gamma : \mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$ is called the *composition product* and $e : I \to \mathcal{P}$ is the *unit* for the composition product.

EXAMPLE. An operad induces a properad as follows

$$\mathsf{Operads} \rightarrowtail \mathsf{Properads}, \quad \mathcal{P} \mapsto \widetilde{\mathcal{P}}, \text{ where } \begin{cases} \widetilde{\mathcal{P}}(1,\, n) & := \quad \mathcal{P}(n) \quad \text{and} \\ \widetilde{\mathcal{P}}(m,\, n) & := \quad 0 \quad\quad \text{for } m \neq 1. \end{cases}$$

Finally, we have the following inclusions:

Monoidal category:	$(\mathbb{K}\text{-}\mathsf{Mod},\,\otimes_{\mathbb{K}})$	$\rightarrowtail$	$(\mathbb{S}\text{-}\mathsf{Mod},\,\circ)$	$\rightarrowtail$	$(\mathbb{S}\text{-}\mathsf{biMod},\,\boxtimes)$
Monoid:	Associative algebras	$\rightarrowtail$	Operads	$\rightarrowtail$	Properads.

The results about properads in this paper apply to algebras and operads as well by the above inclusions of categories.

One defines dually the notions of coalgebra, cooperad, coproperad. For example, a *coproperad* is a comonoid $(\mathcal{C},\,\Delta,\,\eta)$ in the monoidal category of $\mathbb{S}$-bimodules $\mathbb{S}$-biMod. The *coproduct* $\Delta : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$ is coassociative and admits a counit $\eta : \mathcal{C} \to I$. All these definitions extend to the *differential graded* setting, or *dg* setting for short. The differentials are compatible with the properad structure, resp. coproperad structure, in the sense that they are *derivations*, resp. *coderivations* (see [LV] or [Val07] for precise definitions). We will often refer to a dg "object" just as "object," for example we call dg properads "properads."

3. Curved twisting morphisms

In this section, we recall the notion of twisting morphisms for augmented properads and coproperads from [Val08] and [MV09] and the associated bar-cobar adjunction. To extend these notions to the case where the properad is not augmented, we introduce the new notion of *curved coproperad* and of *curved twisting morphism* between a curved coproperad and a not necessarily augmented properad. We also extend the bar and the cobar constructions to this framework. This provides a functorial cofibrant replacement for properads. We emphasize the fact that the properad is not assumed to be augmented.

3.1. **Twisting morphisms.** We recall the theory of twisting morphisms between augmented coproperads and augmented properads from [MV09].

Let M and N be two $\mathbb{S}$-bimodules. By abuse of notation, we will denote by $M \otimes N$ the infinitesimal composite product of one element of M with one element of N grafted above, that is the space of linear combinations of connected graphs with two vertices, the first one labelled by an element of M and the one above labelled by an element of N. This is not quite the same as $\mathcal{Q} \boxtimes_{(1,1)} \mathcal{P}$ of [MV09], in which they define the product of augmented properads, and only take elements from the augmentation ideal. However, we write sometimes $M \boxtimes_{(1,\,1)} N$ instead of $M \otimes N$. To an operad $\mathcal{P}$, we associate the *infinitesimal composition product* $\gamma_{(1,\,1)} : \mathcal{P} \boxtimes_{(1,\,1)} \mathcal{P} \to \mathcal{P}$ with the help of e and γ. Associated to a coproperad $\mathcal{C}$, we define the *infinitesimal decomposition map* $\Delta_{(1,\,1)} : \mathcal{C} \to \mathcal{C} \boxtimes_{(1,\,1)} \mathcal{C}$ by the projection of Δ (with the help of η) on $\mathcal{C} \boxtimes_{(1,\,1)} \mathcal{C}$, or with the above notation, on $\mathcal{C} \otimes \mathcal{C}$.

We recall the convolution product $\star$ on $\mathrm{Hom}(\mathcal{C},\,\mathcal{P}) := \prod_{m,\,n\geq 0} \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}(m,\,n),\,\mathcal{P}(m,\,n))$ from [MV09]. Let $f,\,g \in \mathrm{Hom}(\mathcal{C},\,\mathcal{P})$. We denote by $f \star g$ the composite

$$\mathcal{C} \xrightarrow{\Delta_{(1,\,1)}} \mathcal{C} \boxtimes_{(1,\,1)} \mathcal{C} \xrightarrow{f \boxtimes_{(1,\,1)} g} \mathcal{P} \boxtimes_{(1,\,1)} \mathcal{P} \xrightarrow{\gamma_{(1,\,1)}} \mathcal{P}.$$

We define the derivative ∂ of degree -1 on $\mathrm{Hom}(\mathcal{C},\,\mathcal{P})$ by

$$\partial(f) := d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ d_{\mathcal{C}}.$$

The convolution product $\star$ on $\mathrm{Hom}(\mathcal{C},\,\mathcal{P})$ is a Lie-admissible product (see [MV09] for more details). It is stable on the space of equivariant maps from $\mathcal{C}$ to $\mathcal{P}$ denoted by $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\,\mathcal{P})$. Then the bracket $[f,\,g] := f \star g - (-1)^{|f||g|} g \star f$ is a Lie bracket on $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\,\mathcal{P})$.

A morphism of $\mathbb{S}$-bimodules $\alpha : (\mathcal{C},\,d_{\mathcal{C}}) \to (\mathcal{P},\,d_{\mathcal{P}})$ of degree -1 in the Lie algebra $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\,\mathcal{P})$ is called a *twisting morphism* if it is a solution to the *Maurer-Cartan equation*

$$\partial(\alpha) + \alpha \star \alpha = \partial(\alpha) + \frac{1}{2}[\alpha,\,\alpha] = 0.$$

We denote by $\mathrm{Tw}(\mathcal{C},\,\mathcal{P})$ the set of twisting morphisms in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\,\mathcal{P})$.

We say that an operad $\mathcal{P}$ is *augmented* when there is a morphism $\mathcal{P} \twoheadrightarrow I$ of dg properads such that $I \xrightarrow{e} \mathcal{P} \twoheadrightarrow I$ is the identity. It is equivalent to $\mathcal{P} \cong I \oplus \overline{\mathcal{P}}$ as dg properads where $\overline{\mathcal{P}} := \ker(\mathcal{P} \twoheadrightarrow I)$. Dually, we say that a coproperad $\mathcal{C}$ is *coaugmented* when there is a morphism $I \rightarrowtail \mathcal{C}$ of dg coproperads such that $I \rightarrowtail \mathcal{C} \xrightarrow{\eta} I$ is the identity. It is equivalent to $\mathcal{C} \cong I \oplus \overline{\mathcal{C}}$ as dg coproperads where $\overline{\mathcal{C}} := \mathrm{coker}(I \rightarrowtail \mathcal{C})$. When $\mathcal{P}$ is augmented and $\mathcal{C}$ is coaugmented, we require the twisting morphisms α to satisfy the compositions $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \twoheadrightarrow I$ and $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ being equal to 0. A coaugmented coproperad is called *conilpotent* when for all $x \in \overline{\mathcal{C}}$, there exists an $n > 0$ such that $\overline{\Delta}^n_{(1,1)}(x) = 0$, where $\overline{\Delta}_{(1,1)} : \overline{\mathcal{C}} \to \overline{\mathcal{C}} \boxtimes_{(1,1)} \overline{\mathcal{C}}$ is the primitive part of $\Delta_{(1,1)}$ and where $\overline{\Delta}^n_{(1,1)} = (\overline{\Delta}_{(1,1)} \otimes id_{\mathcal{C}}^{\otimes(n-1)}) \circ \overline{\Delta}^{n-1}_{(1,1)}$ (see [LV] for more details in cooperad case).

When $\mathcal{P}$ is augmented and $\mathcal{C}$ is conilpotent, we recall from [Val07] that the bifunctor $\mathrm{Tw}(-,-)$ is representable on the left by the *cobar construction* and on the right by the *bar construction*, that is we have the following adjunction

$$\Omega : \textsf{conilpotent dg coprop.} \rightleftharpoons \textsf{augmented dg prop.} : \mathrm{B}$$

and there are natural correspondences

$$\mathrm{Hom}_{\textsf{aug. dg prop.}}(\Omega\mathcal{C},\, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C},\, \mathcal{P}) \cong \mathrm{Hom}_{\textsf{conil. dg coprop.}}(\mathcal{C},\, \mathrm{B}\mathcal{P}).$$

3.2. **Curved twisting morphism.** We refine the previous section to the case where $\mathcal{P}$ is not necessarily augmented. A *curvature* has to be introduced on the level of dg coproperads to encode the default of augmentation. The associated notion is called a *curved coproperad.* We define the notion of *curved twisting morphism* between a curved coproperad and a dg properad as a solution of the *curved Maurer-Cartan equation.*

3.2.1. **Curved coproperad.** A *curved coproperad* is a triple $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$, where $\mathcal{C}$ is a graded (but not dg) coproperad, the *predifferential* $d_{\mathcal{C}}$ is a coderivation of $\mathcal{C}$ of degree -1 and the *curvature* $\theta : \mathcal{C} \to I$ is a map of degree -2 such that:

a) $d_{\mathcal{C}}^2 = (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}$,
b) $\theta \circ d_{\mathcal{C}} = 0$.

A *morphism between curved coproperads* $(\mathcal{C},\, d_{\mathcal{C}}, \theta) \to (\mathcal{C}',\, d_{\mathcal{C}'}, \theta')$ is a morphism of coproperads $f : \mathcal{C} \to \mathcal{C}'$ such that $d_{\mathcal{C}'} \circ f = f \circ d_{\mathcal{C}}$ and $\theta' \circ f = \theta$. We denote this category by $\textsf{curved coprop.}$.

We prove the following technical lemma that will be useful later.

3.2.2. **Lemma.** *Let $\mathcal{C}$ be a coproperad. The cobracket $(\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}$ with a linear form $\theta : \mathcal{C} \to I$ is a coderivation.*

PROOF. The coassociativity of $\Delta_{(1,1)}$ gives

$$\begin{aligned}
&\big[\big((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\big) \otimes id_{\mathcal{C}} + id_{\mathcal{C}} \otimes \big((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\big)\big] \circ \Delta_{(1,1)} \\
&\quad = \big[\big((\theta \otimes id_{\mathcal{C}}) \circ \Delta_{(1,1)}\big) \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \big((id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\big)\big] \circ \Delta_{(1,1)} \\
&\quad = (\theta \otimes \Delta_{(1,1)} - \Delta_{(1,1)} \otimes \theta) \circ \Delta_{(1,1)} = \Delta_{(1,1)} \circ \big((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\big).
\end{aligned}$$

□

3.2.3. **The convolution curved Lie algebra.** We define the new notion of *curved Lie algebra* generalizing the notion of dg Lie algebra. A *curved Lie algebra* is a quadruple $(\mathfrak{g},\, [-,\,-],\, d_{\mathfrak{g}},\, \theta)$, where $(\mathfrak{g},\, [-,\,-])$ is a Lie algebra, the predifferential $d_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$ of degree -1 and the curvature θ is an element of $\mathfrak{g}$ (or equivalently a map $\mathbb{K} \to \mathfrak{g}$) of degree -2 such that:

a) $d_{\mathfrak{g}}^2 = [-,\, \theta]$;
b) $d_{\mathfrak{g}}(\theta) = 0$.

Let $(\mathcal{C},\, d_{\mathcal{C}}, \theta)$ be a curved coproperad and let $(\mathcal{P},\, d_{\mathcal{P}})$ be a dg properad. We fix the element

$$\Theta := e \circ \theta : \mathcal{C} \xrightarrow{\theta} I \xrightarrow{e} \mathcal{P}$$

of degree -2 in $\mathrm{Hom}(\mathcal{C},\, \mathcal{P})$.

3.2.4. **Proposition.** *When $\mathcal{C}$ is a curved coproperad and $\mathcal{P}$ is a dg properad, we have on* $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\, \mathcal{P}) = \prod_{m,\, n \geq 0} \mathrm{Hom}_{\mathbb{S}}(\mathcal{C}(m,\, n),\, \mathcal{P}(m,\, n))$*:*

$$\begin{cases} \partial^2 & = \quad [-,\, \Theta] \quad := \quad (\text{-} \star \Theta) - (\Theta \star \text{-}) \\ \partial(\Theta) & = \quad 0. \end{cases}$$

Then $(\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\, \mathcal{P}),\, [-,\,-],\, \partial,\, \Theta)$ *is a curved Lie algebra, called the* convolution curved Lie algebra*.*

PROOF. We do the computations:

$$\begin{aligned}\partial^2(f) &= d_{\mathcal{P}} \circ \partial(f) - (-1)^{|\partial(f)|}\partial(f) \circ d_{\mathcal{C}} \\ &= {d_{\mathcal{P}}}^2 \circ f - (-1)^{|f|} d_{\mathcal{P}} \circ f \circ d_{\mathcal{C}} + (-1)^{|f|}(d_{\mathcal{P}} \circ f \circ d_{\mathcal{C}} - (-1)^{|f|} f \circ {d_{\mathcal{C}}}^2) \\ &= -f \circ {d_{\mathcal{C}}}^2 = -f \circ (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)} = f \star \Theta - \Theta \star f\end{aligned}$$

and $\partial(\Theta) = d_{\mathcal{P}} \circ e \circ \theta - (-1)^{|\Theta|} e \circ \theta \circ d_{\mathcal{C}} = 0$ since $d_{\mathcal{P}} \circ e = 0$ and $\theta \circ d_{\mathcal{C}} = 0$. □

An element $\alpha : (\mathcal{C}, d_{\mathcal{C}}, \theta) \to (\mathcal{P}, d_{\mathcal{P}})$ of degree -1 in the curved Lie algebra $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is called a *curved twisting morphism* if it is a solution of the *curved Maurer-Cartan equation*

$$\partial(\alpha) + \alpha \star \alpha = \Theta.$$

We denote by $\mathrm{Tw}(\mathcal{C}, \mathcal{P})$ the set of curved twisting morphisms in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$.

REMARK. The words “curved” and “curvature” refer to the geometric context. In that setting, the Maurer-Cartan equation applied to a connection provides the curvature form. The flat case corresponds to the curvature equal to zero, that is to the classical case.

3.3. **Bar and cobar constructions.** In this section, we extend the bar construction of augmented dg properads to a *curved bar construction* from dg properads with target in curved coproperads. In the other way round, we extend the cobar construction of coaugmented coproperads to coaugmented curved coproperads. In the algebra case, the cobar construction generalizes the bar construction of curved algebras given in [PP05] and in [Pos93] to properads, though it is not immediate that our constructions are the same, as [PP05, Pos93] do not make use of coalgebras.

3.3.1. **Semi-augmented dg properads.** A *semi-augmented dg properad*, or *sdg properad* for short, $(\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ is a dg properad $\mathcal{P}$ whose underlying $\mathbb{S}$-bimodule is endowed with an augmentation of $\mathbb{S}$-bimodules $\varepsilon : \mathcal{P} \twoheadrightarrow I$, not necessarily dg or of properads, called *semi-augmentation.* In other words, ε is a retraction of $\mathbb{S}$-bimodules of the unit $e : I \to \mathcal{P}$ and we have an isomorphism $e + inc : I \oplus \overline{\mathcal{P}} \xrightarrow{\cong} \mathcal{P}$ of $\mathbb{S}$-bimodules, where $\overline{\mathcal{P}} := \ker \varepsilon$ and inc is the inclusion $\overline{\mathcal{P}} \rightarrowtail \mathcal{P}$. We denote $\rho := (e + inc)^{-1}|^{\overline{\mathcal{P}}} : \mathcal{P} \twoheadrightarrow \overline{\mathcal{P}}$. In the following, we do not write the inclusion inc in the formulae. The map $\overline{\gamma} := \rho \circ \gamma : \overline{\mathcal{P}} \boxtimes \overline{\mathcal{P}} \to \overline{\mathcal{P}}$ is not necessarily associative, even though the composition product $\gamma : \mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$ is associative.

REMARK. The assumption for $\mathcal{P}$ to have a semi-augmentation ε is not restrictive since we are working over a field $\mathbb{K}$ and since we just need to fix a section of $\mathcal{P}(1, 1)$. When $\mathcal{P}(1, 1) = I$, we choose the identity map. This is often the case, as it is for the operad encoding unital associative algebras (see Section 6).

We define on $\overline{\mathcal{P}}$ the map $d_{\overline{\mathcal{P}}} := \rho \circ d_{\mathcal{P}}$, which is a differential since $d_{\mathcal{P}}$ is a differential and since the differential on I is 0. The differentials satisfy $\rho \circ d_{\mathcal{P}} = d_{\overline{\mathcal{P}}} \circ \rho$. However, we have $d_{\overline{\mathcal{P}}} \neq d_{\mathcal{P}}$ in general.

A *morphism between two sdg properads* $(\mathcal{P}, d_{\mathcal{P}}, \varepsilon) \xrightarrow{f} (\mathcal{P}', d_{\mathcal{P}'}, \varepsilon')$ is a morphism of dg properads $f : (\mathcal{P}, d_{\mathcal{P}}) \to (\mathcal{P}', d_{\mathcal{P}'})$ such that $\varepsilon' \circ f = \varepsilon$. We define $\bar{f} := \rho' \circ f : \overline{\mathcal{P}} \to \overline{\mathcal{P}'}$ and we remark that $d_{\overline{\mathcal{P}'}} \circ \bar{f} = \bar{f} \circ d_{\overline{\mathcal{P}}}$. We denote by $\mathsf{sdg\ prop.}$ the category of semi-augmented dg properads.

3.3.2. **Coaugmented and conilpotent curved coproperads.** When $\mathcal{C}$ is coaugmented, that is, $\mathcal{C}$ has a coaugmentation $I \rightarrowtail \mathcal{C}$ so that $\mathcal{C} \cong I \oplus \overline{\mathcal{C}}$ as coproperads, we require that any twisting morphism α satisfies the compositions $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ and $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ to be zero. We denote by $\mathsf{coaug.\ curved\ coprop.}$ the category of coaugmented curved coproperads and by $\mathsf{conil.\ curved\ coprop.}$ the category of conilpotent curved coproperads (see Section 3.1).

We construct a pair of functors

$$\mathrm{B} : \mathsf{sdg\ prop.} \rightleftharpoons \mathsf{coaug.\ curved\ coprop.} : \Omega.$$

Let M be an $\mathbb{S}$-bimodule. The notation $\mathcal{F}(M)$, resp. $\mathcal{F}^c(M)$, stands for the *free properad* on M, resp. the *cofree coproperad* on M. A derivation on $\mathcal{F}(M)$, resp. a coderivation on $\mathcal{F}^c(M)$, is characterized by its restriction on M, resp. by its image on M. The notation sM, resp. $s^{-1}M$, stands for the *homological suspension*, resp. the *homological desuspension*, of the $\mathbb{S}$-bimodule M. We refer to [Val07] for more details.

3.3.3. **Curved bar construction of a sdg properad.** The *bar construction of the sdg properad* $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ is given by the conilpotent curved coproperad

$$\mathrm{B}\mathcal{P} := (\mathcal{F}^c(s\overline{\mathcal{P}}),\, d_{bar},\, \theta_{bar}).$$

The predifferential is defined by $d_{bar} := d_1 + d_2$, where d_2 is the unique coderivation of degree -1 which extends the map

$$\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow \mathcal{F}^c(s\overline{\mathcal{P}})^{(2)} \cong s^2(\overline{\mathcal{P}} \boxtimes_{(1,\,1)} \overline{\mathcal{P}}) \xrightarrow{s^{-1}\overline{\gamma}} s\overline{\mathcal{P}}$$

where $\overline{\gamma} := \rho \circ \gamma : \overline{\mathcal{P}} \boxtimes_{(1,\,1)} \overline{\mathcal{P}} \to \overline{\mathcal{P}}$ and d_1 is the unique coderivation of degree -1 which extends the map

$$\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \xrightarrow{id_s \otimes d_{\overline{\mathcal{P}}}} s\overline{\mathcal{P}}.$$

The curvature θ_{bar} is the map of degree -2

$$\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \oplus \mathcal{F}^c(s\overline{\mathcal{P}})^{(2)} \cong s\overline{\mathcal{P}} \oplus s^2(\overline{\mathcal{P}} \boxtimes_{(1,\,1)} \overline{\mathcal{P}}) \xrightarrow{s^{-1}d_{\mathcal{P}} + s^{-2}\gamma} \mathcal{P} \xrightarrow{\varepsilon} I.$$

3.3.4. **Lemma.** *The predifferential and the curvature satisfy*

a) ${d_{bar}}^2 = (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)}$;
b) $\theta_{bar} \circ d_{bar} = 0$.

Proof. First we can restrict the proof of the equality *a*) and *b*) to $\mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 3)}$ since ${d_{bar}}^2 = \frac{1}{2}[d_{bar},\, d_{bar}]$ and $(\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)}$ are coderivations (see Lemma 3.2.2) and since θ_{bar} is non zero only on $\mathcal{F}^c(s\overline{\mathcal{P}})^{(2)}$.

The composite

$$\mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 3)} \xrightarrow{{d_{bar}}^{|\mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 2)}} - \left[(\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)}\right]^{|I \otimes s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}} \otimes I}} \mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 2)} \oplus$$

$$(I \otimes s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}} \otimes I) \xrightarrow{({d_{bar}}^{|s\overline{\mathcal{P}}} - \theta_{bar}) + \gamma_{|I \otimes s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}} \otimes I}} I \oplus s\overline{\mathcal{P}}$$

equals to $\left({d_{bar}}^2 - (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)} - \theta_{bar} \circ d_{bar}\right)^{|I \oplus s\overline{\mathcal{P}}}$ and to $(d_{\gamma + d_{\mathcal{P}}})^{2|I \oplus s\overline{\mathcal{P}}}$ where $d_{\gamma + d_{\mathcal{P}}}$ is the unique coderivation of degree -1 on $\mathcal{F}^c(s\mathcal{P})$ which extends the map

$$\mathcal{F}^c(s\mathcal{P}) \twoheadrightarrow \overline{\mathcal{F}^c}(s\mathcal{P})^{(\leq 2)} \cong s\mathcal{P} \oplus s^2\mathcal{P} \boxtimes_{(1,\,1)} \mathcal{P} \xrightarrow{id_s \otimes d_{\mathcal{P}} + s^{-1}\gamma} s\mathcal{P}.$$

Moreover, since γ is associative and $d_{\mathcal{P}}$ is a compatible differential, we have ${d_{\gamma + d_{\mathcal{P}}}}^2 = 0$. Thus

$${d_{bar}}^2 - (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)} - \theta_{bar} \circ d_{bar} = 0,$$

that is, due to the degree

$$\begin{cases} {d_{bar}}^2 = (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)} \\ \theta_{bar} \circ d_{bar} = 0. \end{cases}$$

□

3.3.5. **Lemma.** *The bar construction is a functor* B : sdg prop. $\to$ conil. curved coprop..

Proof. Let $f : (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon) \to (\mathcal{P}',\, d_{\mathcal{P}'},\, \varepsilon')$ be a morphism of sdg properads. It induces a morphism of dg $\mathbb{S}$-bimodules $\bar{f} : \overline{\mathcal{P}} \to \overline{\mathcal{P}'}$. The map $\mathcal{F}^c(\bar{f}) : \mathcal{F}^c(s\overline{\mathcal{P}}) \to \mathcal{F}^c(s\overline{\mathcal{P}'})$ is a map of coproperads by construction. The morphism $\bar{f}$ commutes with $\overline{\gamma}_{\mathcal{P}}$ and $\overline{\gamma}_{\mathcal{P}'}$, thus $\mathcal{F}^c(\bar{f})$ commutes with the predifferentials. For a similar reason $\theta'_{bar} \circ \mathcal{F}^c(\bar{f}) = \theta_{bar}$. □

3.3.6. **Cobar construction of a coaugmented curved coproperad.** The *cobar construction of the coaugmented curved coproperad* $(\mathcal{C},\, d_{\mathcal{C}}, \theta)$ is given by the sdg properad

$$\Omega\mathcal{C} := (\mathcal{F}(s^{-1}\overline{\mathcal{C}}),\, d := d_0 + d_1 - d_2,\, \varepsilon).$$

The term d_0 is the unique derivation of degree -1 which extends the map

$$s^{-1}\overline{\mathcal{C}} \xrightarrow{s\theta} I \rightarrowtail \mathcal{F}(s^{-1}\overline{\mathcal{C}}).$$

The term d_1 is the unique derivation of degree -1 which extends the map

$$s^{-1}\overline{\mathcal{C}} \xrightarrow{id_{s^{-1}} \otimes d_{\overline{\mathcal{C}}}} s^{-1}\overline{\mathcal{C}} \rightarrowtail \mathcal{F}(s^{-1}\overline{\mathcal{C}}).$$

The term d_2 is the unique derivation of degree -1 which extends the infinitesimal decomposition map of $\overline{\mathcal{C}}$, up to desuspension:

$$s^{-1}\overline{\mathcal{C}} \xrightarrow{s^{-1}\overline{\Delta}_{(1,1)}} s^{-2}\mathcal{F}^c(\overline{\mathcal{C}})^{(2)} \cong \mathcal{F}(s^{-1}\overline{\mathcal{C}})^{(2)} \rightarrowtail \mathcal{F}(s^{-1}\overline{\mathcal{C}}).$$

The semi-augmentation ε is the natural projection $\mathcal{F}(s^{-1}\overline{\mathcal{C}}) = I \oplus s^{-1}\overline{\mathcal{C}} \oplus \cdots \twoheadrightarrow I$. It is an augmentation of properads but it is not an augmentation of dg properads in general.

3.3.7. **Lemma.** *The derivation d on $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$ satisfies $d^2 = 0$.*

Proof. First of all, if we define the weight on $\mathcal{C}$ by $\mathcal{C}^{(0)} = I$, $\mathcal{C}^{(1)} = \overline{\mathcal{C}}$ and $\mathcal{C}^{(n)} = 0$ when $n \neq 0,\, 1$ and extend it to $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$, we get that the map d_0 is of weight -1, the map d_1 is of weight 0 and the map d_2 is of weight 1. Thus, the term d^2 split in the following way

$$d^2 = \underbrace{{d_0}^2}_{\text{weight}=-2} + \underbrace{d_0 d_1 + d_1 d_0}_{\text{weight}=-1} + \underbrace{{d_1}^2 - d_0 d_2 - d_2 d_0}_{\text{weight}=0} - \underbrace{(d_1 d_2 + d_2 d_1)}_{\text{weight}=1} + \underbrace{{d_2}^2}_{\text{weight}=2}.$$

So, we have to show that each group of terms is equal to zero. The term d_0^2 is zero because $\mathrm{im}(d_0) \subset I$, and any derivation annihilates I. The sum $d_0 d_1 + d_1 d_0$ is zero since $\theta \circ d_{\mathcal{C}} = 0$ and $d_{\mathcal{C}}$ is zero on I and by the Koszul sign rule. The equality $d_{\mathcal{C}}^2 = (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}$ and the Koszul sign rule give ${d_1}^2 - d_0 d_2 - d_2 d_0 = 0$. The equality $d_1 d_2 + d_2 d_1 = 0$ is due to the fact that $d_{\mathcal{C}}$ is a coderivation. Finally ${d_2}^2 = 0$ by "coassociativity" of $\overline{\Delta}_{(1,1)}$ and by the Koszul sign rule. $\square$

3.3.8. **Lemma.** *The cobar construction is a functor* $\Omega : \mathsf{coaug.\ curved\ coprop.} \to \mathsf{sdg\ prop.}$.

Proof. Let $f : (\mathcal{C},\, d_{\mathcal{C}},\, \theta) \to (\mathcal{C}',\, d_{\mathcal{C}'},\, \theta')$ be a morphism between coaugmented curved coproperads. The map $\mathcal{F}(f) : \mathcal{F}(s^{-1}\overline{\mathcal{C}}) \to \mathcal{F}(s^{-1}\overline{\mathcal{C}'})$ is a map of properads by construction and $d_2' \circ \mathcal{F}(f) = \mathcal{F}(f) \circ d_2$ since f is a morphism of coproperads. The equality $d_{\mathcal{C}'} \circ f = f \circ d_{\mathcal{C}}$ implies $d_1' \circ \mathcal{F}(f) = \mathcal{F}(f) \circ d_1$, the equality $\theta' \circ f = \theta$ implies $d_0' \circ \mathcal{F}(f) = \mathcal{F}(f) \circ d_0$ and then $\mathcal{F}(f)$ commutes with the differential. $\square$

3.4. **Bar-cobar adjunction.** The cobar construction on conilpotent curved coproperads and the bar construction on dg properads represent the bifunctor of curved twisting morphisms and form a pair of adjoint functors. The counit of adjunction provides a cofibrant replacement functor for dg properads.

3.4.1. **Theorem.** *For any conilpotent curved coproperad $\mathcal{C}$ and for any sdg properad $\mathcal{P}$, there is are natural correspondences*

$$\mathrm{Hom}_{\mathsf{sdg\ prop.}}(\Omega\mathcal{C},\, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C},\, \mathcal{P}) \cong \mathrm{Hom}_{\mathsf{coaug.\ curved\ coprop.}}(\mathcal{C},\, \mathrm{B}\mathcal{P}).$$

Proof. We make the first bijection explicit. A morphism of sdg properads $f_\alpha : \mathcal{F}(s^{-1}\overline{\mathcal{C}}) \to \mathcal{P}$ is uniquely determined by a map $s\alpha : s^{-1}\overline{\mathcal{C}} \to \mathcal{P}$ of degree 0 such that $s^{-1}\overline{\mathcal{C}} \xrightarrow{s\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ is 0, or equivalently, by a map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree -1 satisfying $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ and $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ are zero (condition for twisting morphisms when $\mathcal{C}$ is coaugmented, see 3.3.2).

Moreover, f_α commutes with the differentials if and only if the following diagram commutes

$$\begin{tikzcd} s^{-1}\overline{\mathcal{C}} \arrow[r, "s\alpha"] \arrow[d, "d_0+d_1-d_2"'] & \mathcal{P} \arrow[r, "d_{\mathcal{P}}"] & \mathcal{P} \\ \mathcal{F}(s^{-1}\overline{\mathcal{C}}) \arrow[rr, "\mathcal{F}(s\alpha)"'] & & \mathcal{F}(\mathcal{P}), \arrow[u, "\widetilde{\gamma}"'] \end{tikzcd}$$

where $\widetilde{\gamma}$ is induced by γ. We have

$$\begin{aligned} & d_{\mathcal{P}} \circ (s\alpha) = s(d_{\mathcal{P}} \circ \alpha) \\ & \widetilde{\gamma} \circ \mathcal{F}(s\alpha) \circ d_0 = e \circ (s\theta) = s(e \circ \theta) \\ & \widetilde{\gamma} \circ \mathcal{F}(s\alpha) \circ d_1 = s\alpha \circ (id_{s^{-1}} \otimes d_{\mathcal{C}}) = -s(\alpha \circ d_{\mathcal{C}}) \\ & \widetilde{\gamma} \circ \mathcal{F}(s\alpha) \circ d_2 = \gamma \circ (s\alpha \boxtimes_{(1,\,1)} s\alpha) \circ s^{-1}\Delta_{(1,\,1)} = s(\gamma \circ (\alpha \boxtimes_{(1,\,1)} \alpha) \circ \Delta_{(1,\,1)}). \end{aligned}$$

Thus the commutativity of the previous diagram is equivalent to the equality

$$e \circ \theta - \alpha \circ d - \gamma \circ (\alpha \boxtimes_{(1,\,1)} \alpha) \circ \Delta_{(1,\,1)} = d_{\mathcal{P}} \circ \alpha,$$

that is $\partial(\alpha) + \alpha \star \alpha = \Theta$.

We now make the second bijection explicit. A morphism of coaugmented coproperads $g_\alpha : \mathcal{C} \to \mathcal{F}^c(s\overline{\mathcal{P}})$ is uniquely determined by a map $s\alpha : \mathcal{C} \to s\overline{\mathcal{P}}$ which sends I to 0, that is by a map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree -1 satisfying $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ and $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ are zero.

Moreover, g_α commutes with the predifferential and with the curvature if and only if the following diagrams commute

$$\begin{tikzcd} \mathcal{C} \arrow[r, "s\alpha+(s\alpha\otimes s\alpha)\circ\Delta_{(1,\,1)}"] \arrow[d, "d_{\mathcal{C}}"'] & s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}} \boxtimes_{(1,\,1)} s\overline{\mathcal{P}} \arrow[d, "d_{bar}=d_1+d_2"] \\ \mathcal{C} \arrow[r, "s\alpha"'] & s\overline{\mathcal{P}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C} \arrow[r, "g_\alpha"] \arrow[d, "\theta"'] & \mathrm{B}\mathcal{P} \arrow[dl, "\theta_{bar}"] \\ I. & \end{tikzcd}$$

Since $\alpha \star \alpha = -(s^{-1}inc) \circ d_2 \circ (s\alpha \otimes s\alpha) \circ \Delta_{(1,\,1)} + e \circ \theta_{bar} \circ g_\alpha$, the commutativity of the diagrams gives $\partial(\alpha) + \alpha \star \alpha = \Theta$. Moreover, the projections of the curved Maurer-Cartan equation on $\overline{\mathcal{P}}$ and on I give the two commutative diagrams. This concludes the proof. ☐

Examples.

- To the identity morphism $id_{\mathrm{B}\mathcal{P}} : \mathrm{B}\mathcal{P} \to \mathrm{B}\mathcal{P}$ of coaugmented curved coproperads corresponds the curved twisting morphism $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ defined by $\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \cong \overline{\mathcal{P}} \rightarrowtail \mathcal{P}$.
- To the identity morphism $id_{\Omega\mathcal{C}} : \Omega\mathcal{C} \to \Omega\mathcal{C}$ of properads corresponds the curved twisting morphism $\iota : \mathcal{C} \to \Omega\mathcal{C}$ defined by $\mathcal{C} \to \overline{\mathcal{C}} \cong s^{-1}\overline{\mathcal{C}} \rightarrowtail \mathcal{F}(s\overline{\mathcal{C}})$.

3.4.2. **Lemma.** *For any conilpotent curved coproperad $\mathcal{C}$ and for any sdg properad $\mathcal{P}$, every curved twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factors through the universal curved twisting morphisms π and ι:*

$$\begin{tikzcd} & \Omega\mathcal{C} \arrow[dr, dashed, "f_\alpha"] & \\ \mathcal{C} \arrow[ur, "\iota"] \arrow[rr, "\alpha"] \arrow[dr, dashed, "g_\alpha"'] & & \mathcal{P} \\ & \mathrm{B}\mathcal{P}, \arrow[ur, "\pi"'] & \end{tikzcd}$$

where f_α is a morphism of sdg properads and g_α is a morphism of conilpotent curved coproperads.

Proof. The dashed arrows are just the images of α by the two bijections of Proposition 3.4.1. ☐

3.4.3. **Weight filtration.** We say that a dg $\mathbb{S}$-bimodule M is *weight filtered differential graded*, or *wfdg* for short, when it is endowed with a filtration of dg $\mathbb{S}$-bimodules $F_\omega M$, $\omega \in \mathbb{N}$. When M is a (co)properad, we assume that the (co)product preserves the filtration. In the weight filtered setting, we only consider those twisting morphisms that preserve the filtration. A wfdg properad $\mathcal{P}$ is called *connected* when $F_0\mathcal{P} = I$ $(= Im(e))$.

We endow any free properad $\mathcal{F}(V)$ with a weight grading given by the number of generators. This induces a weight filtration on any properad $\mathcal{F}(V)/(R)$ defined by generators and relations. Sub-coproperads of $\mathcal{F}^c(V)$ are also weight filtered by the number of generators. When $\mathcal{P}$ is a wfdg properad, $\mathrm{B}\mathcal{P}$ comes equipped with a weight filtration. An element in $\mathrm{B}\mathcal{P}$ is a connected graph whose vertices are labelled by elements μ_i of $\overline{\mathcal{P}}$. It is in the component of weight ω of $\mathrm{B}\mathcal{P}$ if there exist ω_i such that any μ_i is in the component of weight ω_i of $\overline{\mathcal{P}}$ and $\sum \omega_i \leq \omega$. Similarly, we endow $\Omega\mathcal{C}$ with a weight filtering when $\mathcal{C}$ is weight filtered.

The curved twisting morphism π preserves the weight filtration.

3.4.4. **Theorem.** *Let $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ be a connected wfdg semi-augmented properad. The counit of the bar-cobar adjunction is a quasi-isomorphism of wfdg semi-augmented properads, that is the* bar-cobar construction $\Omega\mathrm{B}\mathcal{P}$ *is a resolution of $\mathcal{P}$*

$$\Omega\mathrm{B}\mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

When $\mathcal{P}$ is concentrated in non-negative degree, the bar-cobar construction is a cofibrant properad for the model category defined in Appendix A of [MV07].

Proof. We work in the model category defined in Appendix A of [MV07]. Since $\Omega\mathrm{B}\mathcal{P}$ is quasi-free, the remark after Corollary 40 of [MV07] gives that $\Omega\mathrm{B}\mathcal{P}$ is cofibrant when we assume that $\mathcal{P}$ is non-negatively homologically graded.

As explained in the previous section, $\Omega\mathrm{B}\mathcal{P} = (\mathcal{F}(s^{-1}\overline{\mathcal{F}}(s\overline{\mathcal{P}})),\, d = d_0 + d_1 - d_2)$ is weight filtered by F_p when $\mathcal{P}$ is weight filtered. We have

$$d_0 : F_p \to F_{p-1} \text{ and } d_1 : F_p \to F_p \text{ and } d_2 : F_p \to F_p,$$

where d_0 is induced by θ_{bar}, d_1 is induced by d_{bar} and d_2 is induced by the coproduct on $\mathcal{F}^c(s\overline{\mathcal{P}})$. So F_p is a filtration of chain complexes, it is exhaustive and bounded below and we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

$$E^\bullet_{p,q} \Rightarrow \mathrm{H}_{p+q}(\Omega\mathrm{B}\mathcal{P}).$$

We endow $\mathcal{P}$ with a filtration F'_p induced by the weight. This is a filtration of chain complexes since $d_{\mathcal{P}}$ preserves the weight filtration. The filtration F'_p is exhaustive and bounded below so we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

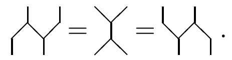
$$E'^\bullet_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P}).$$

The counit of the bar-cobar adjunction preserves the filtration and induces a map of spectral sequences $E^\bullet_{p,q} \to E'^\bullet_{p,q}$. Moreover, $E^0_{\bullet,\bullet} = \Omega\mathrm{B}(gr\mathcal{P})$. The graded properad $gr\mathcal{P}$ associated to the filtration F'_p on $\mathcal{P}$ is always augmented and connected (in the sense of [Val07], that is $gr\mathcal{P}$ is weight graded and $gr\mathcal{P}^{(0)} = I$). However, it is not *reduced*, that is $\mathcal{P}(0,\, n)$ and $\mathcal{P}(m,\, 0)$ can be non zero. Theorem 5.8 of [Val07] applies to reduced properads for which the author provides a canonical writing of an element in $\mathcal{P} \boxtimes \mathcal{P}$ in order to define a contracting homotopy. Such a canonical writing is not possible for non reduced properads. However, it is possible to define a contracting homotopy by means of a sum over all the possibilities. This works for non reduced properads (over a field of characteristic 0) and Theorem 5.8 of [Val07] extends to non reduced properads. So we get that $E^1_{p,q} = gr^{(p)}(\mathrm{H}_{p+q}(gr\mathcal{P}))$. Thus the counit of the bar-cobar adjunction induces an isomorphism of spectral sequences $E^\bullet_{p,q} \to E'^\bullet_{p,q}$ when $\bullet \geq 1$. Since $E'^\bullet_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P})$, the same is true for $E^\bullet_{p,q}$ and the morphism $\Omega\mathrm{B}\mathcal{P} \xrightarrow{\sim} \mathcal{P}$ is a quasi-isomorphism. □

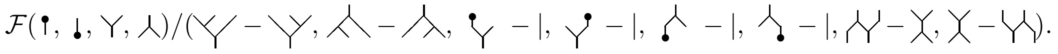
Remarks.

(1) In [Pos09], Positselski defined a bar construction and a cobar construction between curved dg algebras and curved dg coalgebras. The curvatures on both sides encode the default of augmentation or of coaugmentation. In this paper, we are interested only in the default of augmentation and the picture becomes asymmetric. When we reduce our bar construction and our cobar construction to semi-augmented algebras and curved coalgebras, we recover the particular case of [Pos09] where the curved coalgebras are coaugmented.
(2) In [Nic08], Nicolàs proved a similar bar-cobar adjunction on the level of algebras and coalgebras. But the picture is dual. The bar construction goes from *curved associative algebras* to *conilpotent graded-augmented coalgebras* (see [Nic08] for the precise definitions) and the cobar construction goes the other way around. In his case, the curvature does not control the default of augmentation with respect to the composition product and with respect to the dg setting, but only with respect to the dg setting. In the spirit of [Nic08], we should say the dual statement: the default of augmentation with respect to the dg setting measures the curvature.

3.4.5. **Homotopy Frobenius algebras.** A *unital and counital Frobenius algebra* is a quintuple $(A, \mu, \Delta, e, \eta)$ where A is a vector space, $\mu : A \otimes A \to A$ is a commutative and associative product, $\Delta : A \to A \otimes A$ is a cocommutative and coassociative coproduct, $e : \mathbb{K} \to A$ is a unit for the product and $\eta : A \to \mathbb{K}$ is a counit for the coproduct such that the product $\mu =$ and the coproduct $\Delta =$ satisfy the *Frobenius relation*



In operadic terms, we get that A is an algebra over the properad $uc\mathcal{F}rob :=$



This properad is not augmented but Theorem 3.4.4 applies and we get as a corollary:

3.4.6. **Theorem.** *The bar-cobar resolution on $uc\mathcal{F}rob$ is a cofibrant resolution of the properad $uc\mathcal{F}rob$, that is*

$$\Omega \mathrm{B}\, uc\mathcal{F}rob \xrightarrow{\sim} uc\mathcal{F}rob.$$

We define a *$uc\mathcal{F}rob$-algebra up to homotopy* as an algebra over this resolution. As proved in [Abr96, Koc04], the datum of a *2-dimensional topological quantum field theory, 2d-TQFT* for short is equivalent to a unital and counital Frobenius algebra structure. Therefore, we should be able to use this to study 2d-TQFT with homotopy methods.

There is an interesting application in differential geometry. With the present resolution of $uc\mathcal{F}rob$ and with the methods of [Wil07], one endows the differential forms $\Omega_{dR}(M)$ on a closed, oriented manifold M with a structure of $uc\mathcal{F}rob$-algebra up to homotopy, which induces the $uc\mathcal{F}rob$-algebra structure on the cohomology $\mathrm{H}^\bullet(M)$.

4. Curved Koszul duality theory

We extend the Koszul duality theory for homogeneous quadratic properads [Val07] and quadratic-linear properads [GCTV09] to *inhomogeneous quadratic properads* with a quadratic, linear and constant presentation. When the properad is inhomogeneous quadratic, it is not necessarily augmented. Therefore we introduce a Koszul dual coproperad endowed with a curvature, which measures this failure. As explained in Section 2, an associative algebra is a particular kind of properad. Hence this section applies to associative algebras as well to recover the construction given by [Pos93] and [PP05]. However, the presentation given here is slightly different and more general: it works without any finiteness assumption. We end the section with a Poincaré-Birkhoff-Witt theorem for properads.

4.1. **Inhomogeneous quadratic properad.** An *inhomogeneous quadratic properad* is a properad $\mathcal{P}$ which admits a presentation of the form $\mathcal{P} = \mathcal{F}(V)/(R)$, where V is a degree graded $\mathbb{S}$-bimodule and (R) is the ideal generated by a degree graded $\mathbb{S}$-bimodule $R \subset I \oplus V \oplus \mathcal{F}(V)^{(2)}$. The superscript (2) indicates the weight degree. We require that R is a direct sum of (homological) degree homogeneous subspaces. Thus the properad $\mathcal{P}$ is degree graded and has a weight filtration induced by the $\mathbb{S}$-bimodule of generators V. We assume further that the following conditions hold:

(I) The space of generators is minimal, that is $R \cap \{I \oplus V\} = \{0\}$.
(II) The space of relations is maximal, that is $(R) \cap \{I \oplus V \oplus \mathcal{F}(V)^{(2)}\} = R$.

Let $\mathrm{q} : \mathcal{F}(V) \twoheadrightarrow \mathcal{F}(V)^{(2)}$ be the canonical projection and let $\mathrm{q}R \subset \mathcal{F}(V)^{(2)}$ be the image under q of R. We consider the quadratic properad $\mathrm{q}\mathcal{P} := \mathcal{F}(V)/(\mathrm{q}R)$. Since $R \cap \{I \oplus V\} = \{0\}$, there exists a map $\varphi : \mathrm{q}R \to I \oplus V$ such that R is the graph of φ:

$$\begin{aligned} R &= \{X - \varphi(X),\ X \in \mathrm{q}R\} \\ &= \{X - \varphi_1(X) + \varphi_0(X),\ X \in \mathrm{q}R,\ \varphi_1(X) \in V,\ \varphi_0(X) \in \mathbb{K}\}. \end{aligned}$$

The weight grading on the free properad $\mathcal{F}(V)$ induces the following filtration on $\mathcal{P}$

$$F_p := \pi\left(\oplus_{\omega \leq p} \mathcal{F}(V)^{(\omega)}\right),$$

where π stands for the canonical projection $\mathcal{F}(V) \twoheadrightarrow \mathcal{P}$. We denote the associated graded properad by $gr(\mathcal{P})$. The relations $\mathrm{q}R$ hold in $gr(\mathcal{P})$. Therefore, there exists an epimorphism of graded properads

$$p : \mathrm{q}\mathcal{P} \twoheadrightarrow gr(\mathcal{P}).$$

We assume throughout that every inhomogeneous quadratic properad is semi-augmented in the sense of Section 3.3.1. We recall that sV stands for the homological suspension of V, and that the *Koszul dual coproperad of the homogeneous quadratic properad* $\mathrm{q}\mathcal{P}$ is the coproperad cogenerated by sV with corelations in $s^2\mathrm{q}R$ (see Section 2.2 of [Val08]) denoted:

$$\mathrm{q}\mathcal{P}^{¡} := \mathcal{C}(sV,\, s^2\mathrm{q}R) = I \oplus sV \oplus s^2\mathrm{q}R \oplus \cdots .$$

It is a subcoproperad of the cofree coproperad $\mathcal{F}^c(sV)$ on sV. In the cofree coproperad $\mathcal{F}^c(V)$, the weight of an element corresponds to the number of generating elements from V used to write it. There exists a unique coderivation $\tilde{d} : \mathrm{q}\mathcal{P}^{¡} \to \mathcal{F}^c(sV)$ of degree -1 (see Section 3.2 in [MV09]) which extends the map

$$\mathrm{q}\mathcal{P}^{¡} \twoheadrightarrow s^2\mathrm{q}R \xrightarrow{s^{-1}\varphi_1} sV.$$

Moreover, we denote by $\theta : \mathrm{q}\mathcal{P}^{¡} \to I$ the map of degree -2

$$\mathrm{q}\mathcal{P}^{¡} \twoheadrightarrow s^2\mathrm{q}R \xrightarrow{s^{-2}\varphi_0} I.$$

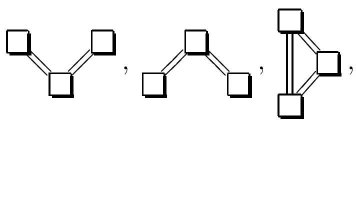
4.1.1. **Lemma.** *Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be an inhomogeneous quadratic properad. Condition (II) implies that:*

- *The coderivation $\tilde{d}$ on $\mathcal{F}^c(sV)$ restricts to a coderivation $d_{\mathcal{P}^{¡}}$ of degree -1 on the subcoproperad $\mathrm{q}\mathcal{P}^{¡} = \mathcal{C}(sV,\, s^2\mathrm{q}R)$;*
- *The coderivation $d_{\mathcal{P}^{¡}}$ satisfies ${d_{\mathcal{P}^{¡}}}^2 = (\theta \otimes id_{\mathrm{q}\mathcal{P}^{¡}} - id_{\mathrm{q}\mathcal{P}^{¡}} \otimes \theta) \circ \Delta_{(1,1)}$;*
- *The coderivation $d_{\mathcal{P}^{¡}}$ satisfies $\theta \circ d_{\mathcal{P}^{¡}} = 0$.*

Proof. We define the map

$$\begin{aligned} \psi :\quad \mathrm{q}R \otimes V \oplus V \otimes \mathrm{q}R &\to \mathcal{F}(V)^{(\leq 3)} \\ r \otimes v + v' \otimes r' &\mapsto (r + \varphi_1(r) - \varphi_0(r)) \otimes v + v' \otimes (r + \varphi_1(r) - \varphi_0(r)). \end{aligned}$$

Since any kind of tree in $\mathcal{F}(V)^{(3)}$ has one of the forms



an element in $\mathrm{q}\mathcal{P}^{\mathrm{i}\,(3)}$ has two decompositions by $\overline{\Delta}_{(1,\,1)}$ in $s^2\mathrm{q}R\otimes sV\oplus sV\otimes s^2\mathrm{q}R\cong s^3(\mathrm{q}R\otimes V\oplus V\otimes \mathrm{q}R)$. Moreover, the two decompositions give the same image with an opposite sign (Koszul sign rule) under ψ. Therefore $\psi\circ(s^{-3}\Delta_{(1,\,1)})(\mathrm{q}\mathcal{P}^{\mathrm{i}\,(3)})\subset\{R\otimes V\oplus V\otimes R\}\cap\{I\oplus V\oplus V^{\otimes 2}\}$.

Condition (II) implies in particular

$$\{R\otimes V+V\otimes R\}\cap\{I\oplus V\oplus V^{\otimes 2}\}\subset R.$$

Projecting on each direct summand, we can rewrite this inclusion as the system of equations

(1) $(s^{-1}\varphi_1\otimes id_{sV}+id_{sV}\otimes s^{-1}\varphi_1)\circ\Delta_{(1,1)}(\mathrm{q}\mathcal{P}^{\mathrm{i}\,(3)})\subset \mathrm{q}\mathcal{P}^{\mathrm{i}\,(2)}$ (projection on $V^{\otimes 2}$);

(2) $\big(s^{-1}\varphi_1\circ(s^{-1}\varphi_1\otimes id_{sV}+id_{sV}\otimes s^{-1}\varphi_1)-(s^{-2}\varphi_0\otimes id_{sV}-id_{sV}\otimes s^{-2}\varphi_0)\big)\circ\Delta_{(1,1)_{|\mathrm{q}\mathcal{P}^{\mathrm{i}\,(3)}}}=0$ (projection on V);

(3) $s^{-2}\varphi_0\circ(s^{-1}\varphi_1\otimes id_{sV}+id_{sV}\otimes s^{-1}\varphi_1)\circ\Delta_{(1,1)_{|\mathrm{q}\mathcal{P}^{\mathrm{i}\,(3)}}}=0$ (projection on I).

By the universal property which defines $\mathrm{q}\mathcal{P}^{\mathrm{i}}=\mathcal{C}(sV,\,s^2\mathrm{q}R)$, it is enough to check that $\tilde{d}(\mathrm{q}\mathcal{P}^{\mathrm{i}\,(3)})\subset \mathrm{q}\mathcal{P}^{\mathrm{i}\,(2)}$ to restrict $\tilde{d}$ to a coderivation of degree -1 on $\mathrm{q}\mathcal{P}^{\mathrm{i}}$, this is exactly the meaning of equation (1). The equation (2) corresponds to the second point of the lemma restricted to $\mathrm{q}\mathcal{P}^{\mathrm{i}\,(3)}$. The equality extends to $\mathrm{q}\mathcal{P}^{\mathrm{i}}$ since ${d_{\mathcal{P}^{\mathrm{i}}}}^2=\frac{1}{2}[d_{\mathcal{P}^{\mathrm{i}}},\,d_{\mathcal{P}^{\mathrm{i}}}]$ and $(\theta\otimes id_{\mathrm{q}\mathcal{P}^{\mathrm{i}}}-id_{\mathrm{q}\mathcal{P}^{\mathrm{i}}}\otimes\theta)\circ\Delta_{(1,1)}$ are coderivations (see Lemma 3.2.2). The equation (3) corresponds to the third point of the lemma since θ is zero outside of $\mathrm{q}\mathcal{P}^{\mathrm{i}\,(2)}$. □

4.2. **Koszul dual coproperad.** Let $\mathcal{P}$ be an inhomogeneous quadratic properad with a quadratic, linear and constant presentation $\mathcal{P}=\mathcal{F}(V)/(R)$ (such that Conditions (I) and (II) hold). The *Koszul dual coproperad* of $\mathcal{P}$ is the weight graded curved coproperad

$$\mathcal{P}^{\mathrm{i}}:=(\mathrm{q}\mathcal{P}^{\mathrm{i}},\,d_{\mathcal{P}^{\mathrm{i}}},\,\theta).$$

4.3. **Koszul properad.** A properad is called a *Koszul properad* if it admits an inhomogeneous quadratic presentation $\mathcal{P}=\mathcal{F}(V)/(R)$ such that Conditions (I) and (II) hold and such that its associated quadratic properad $\mathrm{q}\mathcal{P}:=\mathcal{F}(V)/(\mathrm{q}R)$ is Koszul in the classical sense.

Since the underlying $\mathbb{S}$-bimodule of $\mathcal{P}^{\mathrm{i}}$ is $I\oplus sV\oplus s^2\mathrm{q}R\oplus\cdots$, we define the map of coproperads $g_\kappa:\mathcal{P}^{\mathrm{i}}\rightarrowtail\mathcal{F}^c(sV)\rightarrowtail \mathrm{B}\mathcal{P}$. This map commutes with the predifferentials and with the curvatures, hence it is a morphism of curved coproperads. So by Lemma 3.4.2, there is a curved twisting morphism $\kappa:\mathcal{P}^{\mathrm{i}}\rightarrowtail \mathrm{B}\mathcal{P}\xrightarrow{\pi}\mathcal{P}$. It is explicitly equal to $\mathcal{P}^{\mathrm{i}}\twoheadrightarrow sV\xrightarrow{s^{-1}}V\rightarrowtail\mathcal{P}$. By Theorem 3.4.1, we also obtain a map of dg properads $\Omega\mathcal{P}^{\mathrm{i}}\twoheadrightarrow\Omega\mathrm{B}\mathcal{P}\to\mathcal{P}$.

4.3.1. **Theorem.** *Let $\mathcal{P}$ be a Koszul properad. The cobar construction on the Koszul dual curved coproperad $\mathcal{P}^{i}$ is a cofibrant resolution of $\mathcal{P}$:*

$$\Omega\mathcal{P}^{i}\xrightarrow{\sim}\mathcal{P}.$$

Proof. We work in the model category defined in the Appendix A of [MV07]. Since we are working in the non-negatively graded case and $\Omega\mathcal{P}^{\mathrm{i}}$ is quasi-free, the remark after Corollary 40 gives that $\Omega\mathcal{P}^{\mathrm{i}}$ is cofibrant.

Let $\mathcal{C}:=s^{-1}\overline{\mathrm{q}\mathcal{P}^{\mathrm{i}}}$ be the desuspension of the augmentation coideal of the coproperad $\mathrm{q}\mathcal{P}^{\mathrm{i}}$. So, the underlying $\mathbb{S}$-bimodule of $\Omega\mathcal{P}^{\mathrm{i}}$ is $\mathcal{F}(\mathcal{C})$. Let us consider the new "homological" degree induced by the weight of elements of $\mathrm{q}\mathcal{P}^{\mathrm{i}}$, given by the weight in $\mathcal{F}^c(V)$, minus 1. As in the proof of the Appendix A of [GCTV09], Theorem 30, we call this grading the *syzygy degree*. Therefore, the syzygy degree of an element in $\mathcal{F}(\mathcal{C})$ is given by the sum of the weight of the elements which label its vertices minus the numbers of vertices. Since the weight of an element in $\mathcal{C}$ is greater than 1, the syzygy degree on $\mathcal{F}(\mathcal{C})$ is non-negative.
The term d_0, induced by θ, the term d_1, induced by $d_{\mathcal{P}^{\mathrm{i}}}$ and the term d_2, induced by the infinitesimal decomposition map on $\mathcal{C}$, lower the syzygy degree by 1. Hence, we get a well-defined non-negatively graded chain complex.
We endow $\Omega\mathcal{P}^{\mathrm{i}}=\mathcal{F}(\mathcal{C})$ with a filtration given by the total weight, that is the weight of an element in $\mathcal{F}(\mathcal{C})$ is the sum of the weight of the elements which label the vertices. We have

$$d_0:F_p\to F_{p-2}\text{ and }d_1:F_p\to F_{p-1}\text{ and }d_2:F_p\to F_p.$$

This filtration is exhaustive and bounded below so we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain that

$$E^{\bullet}_{p,\,q} \Rightarrow \mathrm{H}_{p+q}(\Omega\mathcal{P}^{\text{i}}).$$

The filtration F_p induces a filtration F_p on the homology of $\Omega\mathcal{P}^{\text{i}}$ such that

$$E^{\infty}_{p,\,q} \cong F_p(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{\text{i}}))/F_{p-1}(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{\text{i}})) =: gr^{(p)}(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{\text{i}})).$$

Moreover, we have $E^0_{p,\,q} = F_p(\mathcal{F}(\mathcal{C})_{p+q})/F_{p-1}(\mathcal{F}(\mathcal{C})_{p+q}) = \mathcal{F}(\mathcal{C})^{(p)}_{p+q}$, that is the elements of syzygy degree equal to $p+q$ and of weight p. The differential d^0 on the first term of the spectral sequence is given by d_2. Hence, since $\mathrm{q}\mathcal{P}$ is Koszul and concentrated in syzygy degree 0, we have $E^1_{p,\,q} = \mathrm{q}\mathcal{P}^{(p)}_{p+q}$ (Theorem 7.6 of [Val07] by means of the extension seen in the proof of Theorem 3.4.4 applies), concentrated in the line $p+q=0$ and the spectral sequence collapses at rank 1. We have

$$\begin{cases} E^1_{p,\,\text{-}p} &= & \mathrm{q}\mathcal{P}^{(p)} & \cong & E^{\infty}_{p,\,\text{-}p} & \cong & gr^{(p)}(\mathrm{H}_0(\Omega\mathcal{P}^{\text{i}})) \\ E^1_{p,\,q} &= & 0 & = & E^{\infty}_{p,\,q} & \cong & gr^{(p)}(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{\text{i}})) \text{ when } p+q \neq 0. \end{cases}$$

For the syzygy degree, we have

$$\mathrm{H}_0(\Omega\mathcal{P}^{\text{i}}) \cong \mathcal{F}(V)/Im(d_0+d_1-d_2) \cong \mathcal{P}.$$

So, the quotient $gr^{(p)}(\mathrm{H}_0(\Omega\mathcal{P}^{\text{i}}))$ is equal to $gr^{(p)}(\mathcal{P})$. Finally, the morphism $\Omega\mathcal{P}^{\text{i}} \xrightarrow{\sim} \mathcal{P}$ is a quasi-isomorphism. ☐

4.3.2. **Theorem** (Poincaré-Birkhoff-Witt theorem)**.** *When $\mathcal{P}$ is a Koszul properad, the natural epimorphism of properads $\mathrm{q}\mathcal{P} \twoheadrightarrow gr\mathcal{P}$ is an isomorphism of bigraded properads, with respect to the weight grading and the homological degree. Therefore, the following $\mathbb{S}$-bimodules, graded by the homological degree, are isomorphic:*

$$\mathcal{P} \cong gr(\mathcal{P}) \cong \mathrm{q}\mathcal{P}.$$

Proof. It is a direct corollary of the previous proof. ☐

To show Condition (II), that is $(R) \cap \{I \oplus V \oplus \mathcal{F}(V)^{(2)}\} = R$, can be difficult. The following proposition shows that we do not have to compute the full (R) but only the part $\{R \otimes V + V \otimes R\}$.

4.3.3. **Proposition.** *A properad $\mathcal{P}$ is Koszul if and only if it admits a presentation $\mathcal{P} = \mathcal{F}(V)/(R)$ such that $R \subset I \oplus V \oplus \mathcal{F}(V)^{(2)}$ satisfying the following conditions*

(I) $R \cap \{I \oplus V\} = \{0\}$;
(II') $\{R \otimes V + V \otimes R\} \cap \{V \oplus \mathcal{F}(V)^{(2)}\} \subset R$;
(III) *the associated quadratic properad $\mathrm{q}\mathcal{P} := \mathcal{F}(V)/(\mathrm{q}R)$ is Koszul in the classical sense.*

Proof. Definition 4.3 always implies conditions (I), (II') and (III). First, we have to remark that the property (II') instead of (II) is enough to show Lemma 4.1.1 and to define $\mathcal{P}^{\text{i}}$. Moreover, Theorem 4.3.1 and Theorem 4.3.2 are still true. Then we can apply the Poincaré-Birkhoff-Witt Theorem which gives in weight 2 that $\mathrm{q}R = \mathrm{q}((R) \cap \{I \oplus V \oplus \mathcal{F}(V)^{(2)}\})$. This last equality is equivalent to (II) under the condition (I). ☐

4.3.4. **Coloured properad.** Following the ideas of van der Laan in [van03], we can extend this curved Koszul duality to coloured properads. Martin Doubek told us that our construction applies to the coloured operad $\mathcal{I}so$ encoding chain complexes isomorphisms to recover the resolution given by Markl in [Mar01]. Thanks to this resolution, Markl defines a notion of *strong homotopy equivalence* and proves a relax version of the perturbation lemma, that he calls *Ideal Perturbation Lemma.*

5. Resolution of algebras

We now give a resolution of a semi-augmented dg properad $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ as a $\mathcal{P}$-bimodule. In the operadic case, this provides functorial cofibrant resolutions for $\mathcal{P}$-algebras. We use such resolutions to define a cohomology theory associated to unital associative algebras in the next section.

5.1. **Resolutions of properads as bimodule.** We generalize the resolution given by the *bar construction with coefficients* to properads (not necessarily augmented). Moreover, for an inhomogeneous properad which is Koszul, we get a smaller resolution of it called the *Koszul complex.*

5.1.1. **Dg composite product.** Let $(M,\, d_M)$ and $(N,\, d_N)$ be two dg $\mathbb{S}$-bimodules. Recall from [MV07] the differential on the monoidal product $\boxtimes$ of two $\mathbb{S}$-bimodules. Let $id_M \boxtimes' d_N : M \boxtimes N \to M \boxtimes N$ be the morphism of $\mathbb{S}$-bimodules defined by

$$(id_M \boxtimes' d_N)(\rho(m_1,\, \ldots,\, m_b)\sigma(n_1,\, \ldots,\, n_a)\omega) := \sum_{j=1}^{a} \pm\rho(m_1,\, \ldots,\, m_b)\sigma(n_1,\, \ldots,\, d_N(n_j),\, \ldots,\, n_a)\omega$$

and let $d_M \,{}^{\backprime}\!\boxtimes\, id_N : M \boxtimes N \to M \boxtimes N$ be the morphism of $\mathbb{S}$-bimodules defined by

$$(d_M \,{}^{\backprime}\!\boxtimes id_N)(\rho(m_1,\, \ldots,\, m_b)\sigma(n_1,\, \ldots,\, n_a)\omega) := \sum_{i=1}^{b} \pm\rho(m_1,\, \ldots,\, d_M(m_i),\, \ldots,\, m_b)\sigma(n_1,\, \ldots,\, n_a)\omega.$$

This gives a differential on $M \boxtimes N$ by $d_{M\boxtimes N} := d_M \,{}^{\backprime}\!\boxtimes\, id_N + id_M \boxtimes' d_N$.

5.1.2. **Twisted composite product.** In this section, we study the free dg $\mathcal{P}$-bimodules over a curved coproperad $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$. To any map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree $-n$, we associate the unique derivation (see Section 3.2 of [MV07] for precise definitions) of left $\mathcal{P}$-modules $d^l_\alpha : \mathcal{P} \boxtimes \mathcal{C} \to \mathcal{P} \boxtimes \mathcal{C}$ of degree $-n$ which extends the map

$$\mathcal{C} \xrightarrow{\Delta_{(1,\,1)}} \mathcal{C} \boxtimes_{(1,\,1)} \mathcal{C} \xrightarrow{\alpha \otimes id_{\mathcal{C}}} \mathcal{P} \boxtimes_{(1,\,1)} \mathcal{C}.$$

By symmetry, we define also the derivation of right $\mathcal{P}$-modules $d^r_\alpha : \mathcal{C} \boxtimes \mathcal{P} \to \mathcal{C} \boxtimes \mathcal{P}$ of degree $-n$. We endow the free $\mathcal{P}$-bimodule $\mathcal{P} \boxtimes \mathcal{C} \boxtimes \mathcal{P}$ with the following derivation of $\mathcal{P}$-bimodules:

$$d_\alpha := d_{\mathcal{P}\boxtimes\mathcal{C}\boxtimes\mathcal{P}} - d^l_\alpha \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_\alpha,$$

where $d_{\mathcal{P}\boxtimes\mathcal{C}\boxtimes\mathcal{P}} := d_{\mathcal{P}} \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{C}\boxtimes\mathcal{P}} + (id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + id_{\mathcal{P}\boxtimes\mathcal{C}} \boxtimes' d_{\mathcal{P}}$ with $(id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} = id_{\mathcal{P}} \boxtimes' (d_{\mathcal{C}} \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}})$ by associativity of the composite product.

5.1.3. **Lemma.** *On the $\mathcal{P}$-bimodule $\mathcal{P} \boxtimes \mathcal{C} \boxtimes \mathcal{P}$, the derivation d_α satisfies*

$${d_\alpha}^2 = -d^l_{\partial(\alpha)+\alpha\star\alpha-\Theta} \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_{\partial(\alpha)+\alpha\star\alpha-\Theta}.$$

Thus, when $\alpha \in \mathrm{Tw}(\mathcal{C},\, \mathcal{P})$, we have ${d_\alpha}^2 = 0$ and the derivation d_α defines a differential on the chain complex

$$\mathcal{P} \boxtimes_\alpha \mathcal{C} \boxtimes_\alpha \mathcal{P} := (\mathcal{P} \boxtimes \mathcal{C} \boxtimes \mathcal{P}, d_\alpha = d_{\mathcal{P}\boxtimes\mathcal{C}\boxtimes\mathcal{P}} - d^l_\alpha \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_\alpha).$$

Proof. We do the computation for $d_{\mathcal{P}} = 0$, the general case follows immediately. We have

$$\begin{aligned}
{d_\alpha}^2 &= ((id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} - d^l_\alpha \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_\alpha)^2 \\
&= (id_{\mathcal{P}} \boxtimes' d^2_{\mathcal{C}}) \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + (d^l_\alpha)^2 \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' (d^r_\alpha)^2 \\
&\quad -((id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \circ d^l_\alpha + d^l_\alpha \circ (id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}})) \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' ((d_{\mathcal{C}} \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}}) \circ d^r_\alpha + d^r_\alpha \circ (d_{\mathcal{C}} \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}})) \\
&\quad -(d^l_\alpha \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}}) \circ (id_{\mathcal{P}} \boxtimes' d^r_\alpha) - (id_{\mathcal{P}} \boxtimes' d^r_\alpha) \circ (d^l_\alpha \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}}).
\end{aligned}$$

Since ${d_{\mathcal{C}}}^2 = (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,\,1)}$, we have $(id_{\mathcal{P}} \boxtimes' {d_{\mathcal{C}}}^2) \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} = d^l_\Theta \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}} - id_{\mathcal{P}} \boxtimes' d^r_\Theta$. Moreover, the associativity of γ and the coassociativity of $\Delta_{(1,\,1)}$ give $(d^l_\alpha)^2 = -d^l_{\alpha\star\alpha}$ and $(d^r_\alpha)^2 = d^r_{\alpha\star\alpha}$ where the sign is given by the Koszul sign rule and the fact that α has degree -1. Then $(id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \circ d^l_\alpha + d^l_\alpha \circ (id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) = d^l_{\alpha\circ d_{\mathcal{C}}}$ and $(d_{\mathcal{C}} \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}}) \circ d^r_\alpha + d^r_\alpha \circ (d_{\mathcal{C}} \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}}) = d^r_{\alpha\circ d_{\mathcal{C}}}$ since $d_{\mathcal{C}}$ is a coderivation. Finally, $(d^l_\alpha \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}}) \circ (id_{\mathcal{P}} \boxtimes' d^r_\alpha) + (id_{\mathcal{P}} \boxtimes' d^r_\alpha) \circ (d^l_\alpha \,{}^{\backprime}\!\boxtimes\, id_{\mathcal{P}}) = 0$ since α has degree -1. This gives the result. $\square$

5.1.4. **Koszul morphism.** A curved twisting morphism $\alpha : (\mathcal{C}, d_{\mathcal{C}}, \theta) \to (\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ is called a *Koszul morphism* when the map ξ defined by $\mathcal{P} \boxtimes_\alpha \mathcal{C} \boxtimes_\alpha \mathcal{P} \twoheadrightarrow \mathcal{P} \boxtimes I \boxtimes \mathcal{P} \cong \mathcal{P} \boxtimes \mathcal{P} \xrightarrow{\gamma} \mathcal{P}$ is a resolution of $\mathcal{P}$, that is

$$\xi : \mathcal{P} \boxtimes_\alpha \mathcal{C} \boxtimes_\alpha \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

5.1.5. **Proposition.** *Let $\mathcal{P}$ be a wfdg semi-augmented properad. The curved twisting morphism $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ is a curved Koszul morphism, that is, the twisted composite product $\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}$ is a resolution of the properad $\mathcal{P}$ called the* augmented bar resolution

$$\xi : \mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Proof. The method is the same as in the proof of Theorem 3.4.4. The weight filtration on $\mathcal{P}$ induces a filtration on $\mathrm{B}\mathcal{P}$ given by the total weight. This gives a filtration F_p by the weight on $\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}$ and a filtration F'_p by the weight on $\mathcal{P}$. These filtrations are filtrations of chain complexes since the differentials either preserve or decrease the weight. The filtrations are exhaustive and bounded below and the map ξ preserves the filtrations. We apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

$$\begin{cases} E^{\bullet}_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}) \\ E'^{\bullet}_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P}). \end{cases}$$

Since the differential of $E^0_{\bullet,\bullet}$ is the weight preserving part of the differential of $\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}$, the isomorphism of graded vector spaces $E^0_{\bullet,\bullet} \cong gr\mathcal{P} \boxtimes_\pi \mathrm{B}(gr\mathcal{P}) \boxtimes_\pi gr\mathcal{P}$ is an isomorphism of dg modules. Since $gr\mathcal{P}$ is an augmented properad, we can apply Theorem 4.17 of [Val07] (we use the same trick as in the proof of Theorem 3.4.4 for the fact that the properad is non reduced a priori) to $gr\mathcal{P}$ with $R = gr\mathcal{P}$ to get that $E^1_{p,q} = \mathrm{H}_{p+q}(gr^{(p)}\mathcal{P}) = E'^1_{p,q}$. Then $E^r_{p,q}$ and $E'^r_{p,q}$ coincide for $r \geq 1$ and ξ induces an isomorphism between $E^\infty_{p,q}$ and $E'^\infty_{p,q} \cong gr^{(p)}\mathrm{H}_{p+q}(\mathcal{P})$. This concludes the proof. $\square$

Let $\mathcal{P}$ be an inhomogeneous properad, $\mathcal{P}^{¡}$ its Koszul dual cooperad and $\kappa : \mathcal{P}^{¡} \to \mathcal{P}$ the associated curved twisting morphism. The chain complex $\mathcal{P} \boxtimes_\kappa \mathcal{P}^{¡} \boxtimes_\kappa \mathcal{P}$ is called the *total Koszul complex.*

5.1.6. **Proposition.** *Let $\mathcal{P}$ be an inhomogeneous properad and $\mathcal{P}^{i}$ be its Koszul dual coproperad. When $\mathcal{P}$ is Koszul, the curved twisting morphism $\kappa : \mathcal{P}^{i} \to \mathcal{P}$ is a curved Koszul morphism, that is, the total Koszul complex $\mathcal{P} \boxtimes_\kappa \mathcal{P}^{i} \boxtimes_\kappa \mathcal{P}$ is a resolution of the properad $\mathcal{P}$*

$$\xi : \mathcal{P} \boxtimes_\kappa \mathcal{P}^{i} \boxtimes_\kappa \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Proof. The proof is similar to the proof of Proposition 5.1.5. The differences are the following. Since $\mathcal{P}$ is Koszul, the Poincaré-Birkhoff-Witt Theorem 4.3.2 gives $E^0_{\bullet,\bullet} = gr\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{¡} \boxtimes_\kappa gr\mathcal{P} \cong \mathrm{q}\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{¡} \boxtimes_\kappa \mathrm{q}\mathcal{P}$. So the Koszul criterion (Theorem 7.8 of [Val07] with the trick of the proof of Theorem 3.4.4 for the non reduced case) and the comparison Lemma (Theorem 5.4 of [Val07]) with $L = \mathrm{q}\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{¡} \boxtimes_\kappa \mathrm{q}\mathcal{P}$, $L' = \mathrm{q}\mathcal{P}$, $\mathcal{P}' = \mathrm{q}\mathcal{P}$, $M = \mathrm{q}\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{¡}$ and $M' = I$, and the Poincaré-Birkhoff-Witt Theorem 4.3.2 apply to give that $E^1_{p,q} = \mathrm{H}_{p+q}(\mathrm{q}\mathcal{P}^{(p)}) \cong \mathrm{H}_{p+q}(gr^{(p)}\mathcal{P})$. $\square$

5.2. **Resolution of algebras.** From now on, we consider only operads and cooperads since there is in general no notion of free algebra over a properad. In this section, we use the resolutions of $\mathcal{P}$ as a $\mathcal{P}$-bimodule of the previous section to provide functorial resolutions for algebras over $\mathcal{P}$ as, for example, for unital associative algebras (see Section 6).

5.2.1. **Coalgebra over a curved cooperad.** Let $(\mathcal{C}, d_{\mathcal{C}}, \theta)$ be a curved cooperad. A $(\mathcal{C}, d_{\mathcal{C}}, \theta)$-*coalgebra* is a triple (C, Δ_C, d_C) where $(\mathcal{C}, \Delta)$ is a $\mathcal{C}$-coalgebra, and a coderivation $d_C : C \to C$ of degree -1 such that:

$$d_C{}^2 = (\theta \circ id_C) \circ \Delta_C,$$

where the $\circ$ inside the parentheses is the operadic composition product and the $\circ$ outside the parentheses is the composition of morphisms.

A *morphism of $(\mathcal{C}, d_{\mathcal{C}}, \theta)$-coalgebras* $f : (C, \Delta_C, d_C) \to (C', \Delta_{C'}, d_{C'})$ is a morphism $f : C \to C'$ of $\mathcal{C}$-coalgebras which commutes with the predifferentials d_C and $d_{C'}$.

5.2.2. **Relative composition product.** Let $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ be a sdg operad. A *right $\mathcal{P}$-module* $(\mathcal{L},\, \rho)$ is an $\mathbb{S}$-module endowed with a map $\rho : \mathcal{L} \circ \mathcal{P} \to \mathcal{L}$ compatible with the product and the unit of the operad. We define similarly the notion of *left $\mathcal{P}$-module.* We define the relative composite product $\mathcal{L} \circ_{\mathcal{P}} \mathcal{R}$ of a right $\mathcal{P}$-module $(\mathcal{L},\, \rho)$ and a left $\mathcal{P}$-module $(\mathcal{R},\, \lambda)$ by the coequalizer diagram

$$\mathcal{L} \circ \mathcal{P} \circ \mathcal{R} \underset{id_{\mathcal{L}} \circ \lambda}{\overset{\rho \circ id_{\mathcal{R}}}{\rightrightarrows}} \mathcal{L} \circ \mathcal{R} \twoheadrightarrow \mathcal{L} \circ_{\mathcal{P}} \mathcal{R}\ ,$$

where in the above line all $\circ$ are the operadic composition product. These definitions extend to the dg setting.

5.2.3. **Bar construction of $\mathcal{P}$-algebras.** To any curved twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ from a curved cooperad $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$ to an operad $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$, we associate a functor

$$\mathrm{B}_\alpha : \mathsf{dg}\ (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)\text{-}\mathsf{algebras} \to (\mathcal{C},\, d_{\mathcal{C}},\, \theta)\text{-}\mathsf{coalgebras}.$$

For a $\mathcal{P}$-algebra $(A,\, \gamma_A)$, we define on $\mathcal{C}(A) = (\mathcal{C} \circ \mathcal{P}) \circ_{\mathcal{P}} A$ the maps

$$\begin{cases} d_1 : \mathcal{C}(A) \xrightarrow{d_{\mathcal{C}} \circ id_A + id_{\mathcal{C}} \circ' d_A} \mathcal{C}(A) \\ d_2 := d^r_\alpha \circ_{\mathcal{P}} id_A : \mathcal{C}(A) \xrightarrow{\Delta_{(1)} \circ id_A} \mathcal{C} \circ_{(1)} \mathcal{C}(A) \xrightarrow{(id_{\mathcal{C}} \otimes \alpha) \circ id_A} \mathcal{C} \circ \mathcal{P}(A) \xrightarrow{id_{\mathcal{C}} \circ \gamma_A} \mathcal{C}(A), \end{cases}$$

where $(\mathcal{C} \circ \mathcal{P}) \circ A \xrightarrow{d^r_\alpha \circ id_A} (\mathcal{C} \circ \mathcal{P}) \circ A \twoheadrightarrow \mathcal{C}(A)$ factors through $\mathcal{C}(A)$ to give $d^r_\alpha \circ_{\mathcal{P}} id_A$ since γ_A is a dg map. (Here, $\Delta_{(1)}$ corresponds to the infinitesimal decomposition map $\Delta_{(1,1)}$ and $\mathcal{C} \circ_{(1)} \mathcal{C}$ corresponds to $\mathcal{C} \boxtimes_{(1,1)} \mathcal{C}$ when we restrict to cooperads.)

5.2.4. **Lemma.** *Since α is a curved twisting morphism, we have*

$$(d_1 + d_2)^2 = (\theta \circ id_{\mathcal{C}(A)}) \circ \Delta_{\mathcal{C}(A)}.$$

Proof. We compute

$$\begin{cases} \quad {d_1}^2 & = {d_{\mathcal{C}}}^2 \circ id_A = ((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1)}) \circ id_A \\ & = (\theta \circ id_{\mathcal{C}(A)}) \circ (\Delta_{(1)} \circ id_A) - d^r_\Theta \circ_{\mathcal{P}} id_A \\ & = (\theta \circ id_{\mathcal{C}(A)}) \circ \Delta_{\mathcal{C}(A)} - d^r_\Theta \circ_{\mathcal{P}} id_A \qquad\qquad (\theta \text{ is non-zero only on } \mathcal{P}(1)) \\ \quad {d_2}^2 & = d^r_{\alpha \star \alpha} \circ_{\mathcal{P}} id_A \\ d_1 d_2 + d_2 d_1 & = d^r_{\partial(\alpha)} \circ_{\mathcal{P}} id_A. \end{cases}$$

Thus $(d_1 + d_2)^2 = d^r_{\partial(\alpha) + \alpha\star\alpha - \Theta} \circ_{\mathcal{P}} id_A + (\theta \circ id_{\mathcal{C}(A)}) \circ \Delta_{\mathcal{C}(A)}$ and we get the result since α is a curved twisting morphism. ☐

The *bar construction on A* is the $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$-coalgebra $\mathrm{B}_\alpha A := (\mathcal{C}(A),\, d := d_1 + d_2)$.

5.2.5. **Cobar construction of a $\mathcal{C}$-coalgebra.** Similarly to the previous section, to any curved twisting morphism $\alpha : (\mathcal{C},\, d_{\mathcal{C}},\, \theta) \to (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$, we associate a functor

$$\Omega_\alpha : (\mathcal{C},\, d_{\mathcal{C}},\, \theta)\text{-}\mathsf{coalgebras} \to \mathsf{dg}\ (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)\text{-}\mathsf{algebras}.$$

For any $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$-coalgebra $(C,\, \Delta_C,\, d_C)$, we define on $\mathcal{P}(C)$ the maps

$$\begin{cases} d_1 : \mathcal{P}(C) \xrightarrow{d_{\mathcal{P}} \circ id_C + id_{\mathcal{P}} \circ' d_C} \mathcal{P}(C) \\ d_2 : \mathcal{P}(C) \xrightarrow{id_{\mathcal{P}} \circ' \Delta_C} \mathcal{P} \circ_{(1)} \mathcal{C}(C) \xrightarrow{(id_{\mathcal{P}} \otimes \alpha) \circ id_C} \mathcal{P} \circ \mathcal{P}(C) \xrightarrow{\gamma \circ id_C} \mathcal{P}(C). \end{cases}$$

5.2.6. **Lemma.** *Since α is a curved twisting morphism, we have*

$$(d_1 - d_2)^2 = 0.$$

Proof. We compute

$$\begin{cases} \quad {d_1}^2 & = id_{\mathcal{P}} \circ' {d_C}^2 = id_{\mathcal{P}} \circ' ((\theta \circ id_C) \circ \Delta_C) \\ \quad {d_2}^2 & = -(\gamma \circ id_C) \circ (id_{\mathcal{P}} \circ (\alpha \star \alpha) \circ id_C) \circ (id_{\mathcal{P}} \circ' \Delta_C) \\ -d_1 d_2 - d_2 d_1 & = -(\gamma \circ id_C) \circ (id_{\mathcal{P}} \circ \partial(\alpha) \circ id_C) \circ (id_{\mathcal{P}} \circ' \Delta_C). \end{cases}$$

Thus $(d_1 - d_2)^2 = -(\gamma \circ id_C) \circ (id_{\mathcal{P}} \circ (\partial(\alpha) + \alpha \star \alpha - \Theta) \circ id_C) \circ (id_{\mathcal{P}} \circ' \Delta_C) = 0$ since α is a curved twisting morphism. ☐

The *cobar construction on C* is the dg $\mathcal{P}$-algebra $\Omega_\alpha C := (\mathcal{P}(C),\, d_{\Omega_\alpha C} := d_1 - d_2)$.

5.2.7. **The bar-cobar resolution.** The bar-cobar construction on a $\mathcal{P}$-algebra provides a functorial cofibrant resolution of any $\mathcal{P}$-algebra when the curved twisting morphism α is Koszul.

5.2.8. **Proposition.** *Let $\alpha : (\mathcal{C}, d_{\mathcal{C}}, \theta) \to (\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ be a curved Koszul morphism between a curved cooperad $(\mathcal{C}, d_{\mathcal{C}}, \theta)$ and a sdg operad $(\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ which are bounded below. Then the bar-cobar resolution $\Omega_\alpha \mathrm{B}_\alpha A$ is a resolution of the $\mathcal{P}$-algebra A, that is,*

$$\Omega_\alpha \mathrm{B}_\alpha A = \mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A \xrightarrow{\sim} A.$$

Moreover when A is bounded below, it is a cofibrant resolution.

Proof. There is a model category structure on the category of right $\mathcal{P}$-modules given in Proposition 14.1.A of [Fre09]. The cofibrant objects are described in Proposition 14.2.2 of [Fre09] and since the cooperad $\mathcal{C}$ and the operad $\mathcal{P}$ are bounded below, the right $\mathcal{P}$-module $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}$ is cofibrant. Finally, Theorem 15.1.A of [Fre09] gives that $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A \cong (\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ_{\mathcal{P}} A \xrightarrow{\sim} \mathcal{P} \circ_{\mathcal{P}} A \cong A$ is a resolution.

In the (semi-)model category structure on $\mathcal{P}$-algebras defined in [Fre09], cofibrant $\mathcal{P}$-algebras are retracts of quasi-free $\mathcal{P}$-algebras endowed with a good filtration (Proposition 12.3.8 in [Fre09]). This is the case here since the chain complexes are bounded below. □

5.2.9. **Theorem.** *Let $(\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ be a sdg operad. The curved Koszul morphism $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ gives a resolution*

$$\Omega_\pi \mathrm{B}_\pi A = \mathcal{P} \circ_\pi \mathrm{B}\mathcal{P} \circ_\pi A \xrightarrow{\sim} A,$$

which is cofibrant when A is bounded below. When $\mathcal{P}$ is a Koszul operad, the total Koszul complex gives a smaller resolution

$$\Omega_\kappa \mathrm{B}_\kappa A = \mathcal{P} \circ_\kappa \mathcal{P}^{i} \circ_\kappa A \xrightarrow{\sim} A,$$

which is cofibrant when A is bounded below.

Proof. It is a direct corollary of Proposition 5.2.8 and Propositions 5.1.5 and 5.1.6. □

6. Homotopy and cohomology theories for unital associative algebras

In this section we describe a simple resolution of the operad which encodes unital associative algebras, $u\mathcal{A}s$, obtained by the methods described in section 4. In fact, many of the theorems in this section can be generalized in a straightforward way to any (inhomogeneous) Koszul properad. Algebras over the resolution uA_∞ are called *homotopy unital A_∞-algebras*, or *uA_∞-algebras*, for short. We use some nice properties of our resolution to prove that uA_∞-algebras may be replaced up to equivalence by strictly unital associative algebras. Using our explicit transfer formulae, we show that a unital associative algebra may be transferred to homology as a strictly unital A_∞-algebra (see Definition 6.5.1). This gives a proof that one may always choose a minimal model for a uA_∞-algebra which is actually a strictly unital A_∞-algebra. In this sense, it is "enough" to resolve only the associative relation of $u\mathcal{A}s$, obtaining the operad A_∞, and then adjoin a unit, giving the operad which encodes strictly unital A_∞-algebras. As a corollary of our discussion, we provide sufficient conditions so that: "When trying to find resolutions of algebraic structures with units, it is 'good enough' to resolve the structure (without its units) first, and then append the units to that resolution." The notion of uA_∞-algebras is exactly the notion of "A_∞-algebras with a homotopy unit" of [FOOO09]. Concerning the notion of ∞-morphism and the nice properties of uA_∞-algebras, we still have to compare them with the theory presented in [FOOO09].

6.1. **Homotopy unital associative algebras.** We give a presentation for the operad encoding unital associative algebras. This presentation is an inhomogeneous quadratic presentation and we can apply the theory of the previous sections to compute its Koszul dual cooperad, and hence an explicit resolution.

We use the notation $\underline{n} := \{1, \ldots, n\}$. The symbol $\overline{\mu}$ stands for an element in a cooperad and the symbol μ stands for an element in an operad.

6.1.1. **The operad encoding unital associative algebras.** We denote by $u\mathcal{A}s$ the operad whose representations in the category of dg modules are precisely differential graded unital associative algebras. We consider the following presentation

$$u\mathcal{A}s = \mathcal{F}\,(\ ,\ )\,/(\ -\ ,\ \ -\ |,\ \ -\ |).$$

Remark. We fix this presentation to make our computations of the Koszul dual, $u\mathcal{A}s^{\text{¡}}$ and ultimately uA_∞. Note that this presentation for $u\mathcal{A}s$ is an inhomogeneous quadratic presentation (see 4.1 for a definition).

To make the Koszul dual cooperad, $u\mathcal{A}s^{\text{¡}}$ of $u\mathcal{A}s$ explicit, we compute its associated quadratic operad:

$$\mathrm{q}u\mathcal{A}s^{\text{¡}} = \mathcal{F}(\ ,\ )/(\ -\ ,\ \ ,\ \ ) = \ \oplus \mathcal{A}s.$$

Let's take a moment to explain the notation on the right-hand side of the equation above.

6.1.2. **Definition.** *Let $\mathcal{P}$, $\mathcal{Q}$ be augmented operads. Then the* direct sum operad $\mathcal{P} \oplus \mathcal{Q}$ *is defined to be $\mathcal{F}\left(\overline{\mathcal{P}}, \overline{\mathcal{Q}}\right) / \left(R_{\mathcal{P}}, R_{\mathcal{Q}}, R_{\mathcal{P}\mathcal{Q}}\right)$, where $R_{\mathcal{P}}, R_{\mathcal{Q}}$ are the relations in $\mathcal{P}, \mathcal{Q}$ respectively, and $R_{\mathcal{P}\mathcal{Q}}$ is the collection of all compositions of a pair of elements, one in $\overline{\mathcal{P}}$, one in $\overline{\mathcal{Q}}$.*

Remark. The direct sum operad is the product in the category of augmented operads.

6.1.3. **Proposition.** *If $\mathcal{P}$ and $\mathcal{Q}$ are both quadratic augmented operads, then $\mathcal{P} \oplus \mathcal{Q}$ is a quadratic augmented operad.*

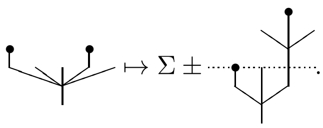
Proof. For any two presented operads, $\mathcal{P} = \mathcal{F}(V_1)/(R_1)$, $\mathcal{Q} = \mathcal{F}(V_2)/(R_2)$, the direct sum operad $\mathcal{P} \oplus \mathcal{Q}$ is naturally presented by $\mathcal{F}(V_1, V_2)/(R_1, R_2, R_{V_1V_2})$. If (V_1, R_1) and (V_2, R_2) are both quadratic presentations, then so is the natural presentation for $\mathcal{P} \oplus \mathcal{Q}$. $\square$

We will make use of the identification $\mathrm{q}u\mathcal{A}s^{\text{¡}} = \ \oplus \mathcal{A}s$ to compute the Koszul dual cooperad of $\mathrm{q}u\mathcal{A}s^{\text{¡}}$ (see 6.1.4). Before we compute the resulting cooperad, $\mathrm{q}u\mathcal{A}s^{\text{¡}}$, we first describe it.

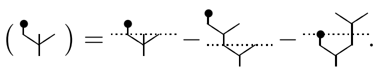
Linearly, we have an isomorphism $\mathrm{q}u\mathcal{A}s^{\text{¡}} \cong \mathbb{K}\left[\overline{\mu}_n^S\right]_{n\geq 1, S\subset \underline{n}}$. The element $\overline{\mu}_n^S \in \mathrm{q}u\mathcal{A}s^{\text{¡}}$ corresponds to a (co)operation with $n - |S|$ inputs: however, we draw this operation as a corolla with n leaves, and a *cork* covering each of the leaves in the set S. For example, $\overline{\mu}_5^{\{1,4\}}$ corresponds to . We point out here that the space of n-to-1 operations is infinite dimensional for every $n \geq 0$. To see this, note that every n-to-1 corolla is an n-ary operation, and by adding a corked leaf, we get a new n-to-1 operation. Continuing to add corked leaves gives infinitely many new n-ary operations.

Also notice that $\overline{\mu}_n^{\emptyset} =$ for $n \geq 1$ spans the subcooperad corresponding to $\mathcal{A}s^{\text{¡}}$ and $\left\{\overline{\mu}_1^{\emptyset} = |, \overline{\mu}_1^{\{1\}} = \ \right\}$ spans the subcooperad corresponding to $^{\text{¡}}$ (with $\overline{\mu}_1^{\emptyset}$ corresponding to the identity cooperation in both cases).

Using this basis, the infinitesimal decomposition $\Delta_{(1)}$ is given by summing over all possible (nontrivial) ways to split the corolla into two, preserving the number of leaves and the number and positions of the corks. Pictorially:



For example,

$\Delta_{(1)}$ 

We compute the Koszul dual cooperad, $\mathrm{q}u\mathcal{A}s^{\text{¡}}$ by the following proposition.

6.1.4. **Proposition.** *Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be a quadratic operad where V is finite-dimensional. Then by Proposition 6.1.3 the operad $\ \oplus \mathcal{P}$ is given by $(\ \oplus \mathcal{P})(0) := (\mathbb{K}\cdot\ ) \oplus \mathcal{P}(0)$ and $(\ \oplus \mathcal{P})(n) := \mathcal{P}(n)$ for all $n \neq 0$ and endowed with the operadic structure given by the (trivial) structure on , the* *structure on $\mathcal{P}$, and trivial composition between $\dagger$ and $\mathcal{P}$. The Koszul dual cooperad of $\dagger \oplus \mathcal{P}$ is given by the coaugmented cooperad*

$$(\dagger \oplus \mathcal{P})^{i} \cong \mathbb{K} \cdot \{\overline{\mu}_S, \text{ where } \overline{\mu} \in \mathcal{P}^{i}(n),\ S \subset \underline{n} \text{ and } |\overline{\mu}_S| = |\overline{\mu}| + |S|\}.$$

The set S is the set of the positions of the "corks" $\dagger$. Let $\overline{\xi} \in \mathcal{P}^{i}(n)$ such that $\Delta_{(1)}(\overline{\xi}) = \sum(\overline{\mu}; \underbrace{id, \ldots, id}_{p}, \overline{\nu}, \underbrace{id, \ldots, id}_{r})$, where $\overline{\mu} \in \mathcal{P}^{i}(m)$, $\overline{\nu} \in \mathcal{P}^{i}(q)$, $p+1+r = m$ and $p+q+r = n$. Then the infinitesimal decomposition map on $\overline{\xi}_S \in (\dagger \oplus \mathcal{P})^{i}$, where $S \subset \underline{n}$, is given by

$$\Delta_{(1)}(\overline{\xi}_S) = \sum (-1)^{\epsilon} (\overline{\mu}_{S_1}; \underbrace{id, \ldots, id}_{p-|S_1'|}, \overline{\nu}_{S_2}, \underbrace{id, \ldots, id}_{r-|S_1''|}),$$

where $\epsilon = |\overline{\nu}||S_1| + |S_2||S_1''|$,

$\overline{\mu}_{S_1} \in \mathcal{P}^{i}(m-|S_1|)$, $\overline{\nu}_{S_2} \in \mathcal{P}^{i}(q-|S_2|)$ and $\left\{\begin{array}{lll} S_1' & \subset & \underline{p} \\ S_2 & \subset & \underline{q} \\ S_1'' & \subset & \{p+2, \ldots, p+1+r\} \end{array}\right.$ such that $S = S_1' \sqcup (S_2 + p) \sqcup (S_1'' + q - 1)$ and $S_1 = S_1' \sqcup S_1''$.

Proof. The operad $\dagger \oplus \mathcal{P}$ is a quadratic operad given by $\mathcal{F}(\dagger \oplus V)/(R \oplus V \otimes \dagger)$ where

$$V \otimes \dagger := \{\mu^{\{k\}}, \text{ with } \mu \in V(n) \text{ and } \{k\} \subset \underline{n}\}.$$

We follow Appendix B of [Lod01] defining $\mathcal{P}^! := \mathcal{F}(V^\vee)/(R^\perp)$, where $V^\vee := V^* \otimes (sgn)$ with the signature representation (sgn) and $R^\perp$ is the orthogonal space for the natural pairing $\langle -, - \rangle : V^\vee \otimes V \to \mathbb{K}$. Since $(V \otimes \dagger)^\perp = \mathcal{F}_{(2)}(V^\vee)$, we get

$$(\dagger \oplus \mathcal{P})^! = \mathcal{F}(\dagger^\vee \oplus V^\vee)/(R^\perp \cap \mathcal{F}_{(2)}(V^\vee)) \cong \{\mu^S, \text{ where } \mu \in \mathcal{P}^!(n) \text{ and } S \subset \underline{n}\}$$

and the composition is induced, up to signs, by the composition on $\mathcal{P}^!$.

Following [LV], we have $\mathcal{P}^{¡} := \mathcal{S}^{-1^c} \otimes_H (\mathcal{P}^!)^*$ where $\mathcal{S}^{-1^c}$ is the operadic desuspension. Then the Koszul dual cooperad of $\dagger \oplus \mathcal{P}$ is equal to

$$(\dagger \oplus \mathcal{P})^{¡} \cong \{\overline{\mu}^S, \text{ where } \overline{\mu} \in \mathcal{P}^{¡}(n),\ S \subset \underline{n} \text{ and } |\overline{\mu}^S| = |\overline{\mu}| + |S|\}$$

and the (infinitesimal) cocomposition is given, up to signs, by the (infinitesimal) cocomposition of $\mathcal{P}^{¡}$. To compute the signs, we recall that the corks $\dagger$ have degree -1 and we apply the Koszul rule. The sign $(-1)^{|\overline{\nu}||S_1|}$ in the formula of the proposition comes from the fact that $\overline{\nu}$ passes through the corks indexed by S_1 and the sign $(-1)^{|S_2||S_1''|}$ comes from the fact that the corks indexed by S_2 pass through the corks indexed by S_1''. ☐

6.1.5. **Corollary.** *The Koszul dual cooperad associated to* qu$\mathcal{A}s$ *is equal to*

$$\mathrm{qu}\mathcal{A}s^{i} = (\dagger \oplus \mathcal{A}s)^{i} \cong \mathbb{K}[\overline{\mu}_n^S],$$

where $\overline{\mu}_n \in \mathcal{A}s^{i}(n)$, $S \subset \underline{n}$, so $\overline{\mu}_n^S \in u\mathcal{A}s^{i}(n-|S|)$ and $|\overline{\mu}_n^S| = n-1+|S|$. The infinitesimal decomposition map is given by

$$\Delta_{(1)}(\overline{\mu}_n^S) = \sum_{\substack{p+q+r=n \\ p+1+r=m}} (-1)^{(q+1)(r+|S_1|)+|S_2||S_1''|} (\overline{\mu}_m^{S_1}; \underbrace{id, \ldots, id}_{p-|S_1'|}, \overline{\mu}_q^{S_2}, \underbrace{id, \ldots, id}_{r-|S_1''|}),$$

where $\left\{\begin{array}{lll} S_1' & \subset & \underline{p} \\ S_2 & \subset & \underline{q} \\ S_1'' & \subset & \{p+2, \ldots, p+1+r\} \end{array}\right.$ such that $S = S_1' \sqcup (S_2 + p) \sqcup (S_1'' + q - 1)$ and $S_1 = S_1' \sqcup S_1''$.

Moreover, the coproduct is given by

$$\Delta(\overline{\mu}_n^S) = \sum_{i_1 - |T_1| + \cdots + i_{m-|T|} - |T_{m-|T|}| = n - |S|} (-1)^{\epsilon} (\overline{\mu}_m^T; \overline{\mu}_{i_1}^{T_1}, \ldots, \overline{\mu}_{i_{m-|T|}}^{T_{m-|T|}}),$$

where $\left\{\begin{array}{lcl} T & \subset & \underline{m} \\ T_j & \subset & \underline{i_j} \end{array}\right.$ *such that* $T = R_0 \sqcup \ldots \sqcup R_{m-|T|}$ *and*

$$\begin{array}{rcl} S & = & R_0 \sqcup (T_1 + |R_0|) \sqcup (R_1 + i_1) \sqcup \ldots \sqcup \\ & & (T_{m-|T|} + |R_0| + \cdots + |R_{m-|T|-1}| + i_1 + \cdots + i_{m-|T|-1}) \sqcup (R_{m-|T|} + i_1 + \cdots + i_{m-|T|}) \end{array}$$

and where

$$\epsilon := |T|(n-m) + \sum_{j=1}^{m-|T|} \big[(i_j - 1)(k - j + |T_1| + \cdots + |T_{j-1}|) + |R_j|(|T_1| + \cdots + |T_j|)\big].$$

PROOF. Provided that the degree of $\overline{\mu}_n \in \mathcal{A}s^{\text{i}}(n)$ is $n-1$ and provided the formula for the coproduct in $\mathcal{A}s^{\text{i}}$ given in [LV], chapter 8, where we include the decomposition involving $\overline{\mu}_m = |$ or $\overline{\mu}_q = |$,

$$\Delta_{(1)}(\overline{\mu}_n) = \sum_{\substack{p+q+r=n \\ p+1+r=m}} (-1)^{(q+1)r} (\overline{\mu}_m; \underbrace{\text{id}, \ldots, \text{id}}_{p}, \overline{\mu}_q, \underbrace{\text{id}, \ldots, \text{id}}_{r}),$$

Proposition 6.1.4 gives the description of $\text{q}u\mathcal{A}s^{\text{i}}$ and of the infinitesimal decomposition map. The coproduct is given in the same way as explained in the proof of Proposition 6.1.4 thanks to the coproduct in $\mathcal{A}s^{\text{i}}$ given in [LV] by

$$\Delta(\overline{\mu}_n) = \sum_{i_1 + \cdots + i_m = n} (-1)^{\epsilon'} (\overline{\mu}_m; \overline{\mu}_{i_1}, \ldots, \overline{\mu}_{i_m}),$$

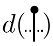
where $\epsilon' := \sum_{j=1}^{m} (i_j - 1)(k - j)$. $\square$

6.1.6. **Proposition.** *The operad* $\text{q}u\mathcal{A}s$ *is Koszul, that is*

$$\text{q}u\mathcal{A}s \circ_\kappa \text{q}u\mathcal{A}s^{i} \xrightarrow{\sim} I.$$

PROOF. We remark that

$$\text{q}u\mathcal{A}s \circ_\kappa \text{q}u\mathcal{A}s^{\text{i}} \cong$$ $$\oplus \left\{ \bigoplus_{S \subseteq \underline{n}} (\mathcal{A}s \circ_\kappa \mathcal{A}s^{\text{i}})(n) \right\}_{n \geq 1}.$$

Since  $= 0 =$  in $\text{q}u\mathcal{A}s$, the differential on $(\mathcal{A}s \circ_\kappa \mathcal{A}s^{\text{i}})(n)$ is given by the usual differential on $\mathcal{A}s \circ_\kappa \mathcal{A}s^{\text{i}}$ except for $d($$) =$. Moreover, we know that $(\mathcal{A}s \circ_\kappa \mathcal{A}s^{\text{i}})(n) \xrightarrow{\sim} I(n)$. Thus $\text{q}u\mathcal{A}s \circ_\kappa \text{q}u\mathcal{A}s^{\text{i}} \xrightarrow{\sim} I$. $\square$

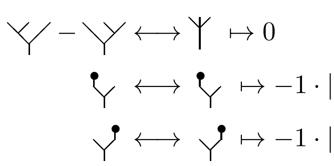
6.1.7. **Lemma.** *The curved cooperad* $u\mathcal{A}s^{i}$ *is equal to the curved cooperad*

$$u\mathcal{A}s^{i} = (\text{q}u\mathcal{A}s^{i}, \Delta_{\text{q}u\mathcal{A}s^{i}}, 0, \theta),$$

where $\Delta_{\text{q}u\mathcal{A}s^{i}}$ *was made explicit in Corollary 6.1.5 and*

$$\theta(\mu_n^S) = \left\{ \begin{array}{ll} -1 \cdot | & \textit{if } n = 2 \textit{ and, } S = \{1\} \textit{ or } S = \{2\} \\ 0 & \textit{otherwise} \end{array} \right. .$$

PROOF. For the definitions given in 4.1, we remark that the space of generators defining $u\mathcal{A}s$ satisfies Conditions (I) and (II) of Section 4.1. According to the definition 4.2, we just have to compute the predifferential $d_{u\mathcal{A}s^{\text{i}}}$ and its curvature θ. Since the relations in $u\mathcal{A}s$ have no linear terms, the predifferential $d_{u\mathcal{A}s^{\text{i}}} = 0$. To compute θ, we find the elements of weight 2, which correspond to the relations in $\text{q}u\mathcal{A}s$. We identify each cooperation with the corresponding leading quadratic term of a relation in $u\mathcal{A}s$, and then assign to that operation the opposite of the corresponding constant term of the relation:



$\square$

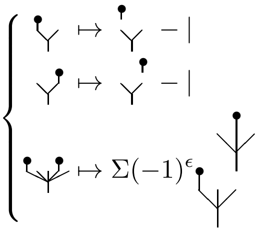
6.1.8. **Theorem.** *The cobar construction on the Koszul dual curved cooperad associated to $u\mathcal{A}s$ provides a cofibrant resolution of $u\mathcal{A}s$*

$$uA_\infty := \Omega u\mathcal{A}s^i \xrightarrow{\sim} u\mathcal{A}s.$$

Proof. By Proposition 6.1.6, $\mathrm{q}u\mathcal{A}s$ is Koszul, and then Theorem 4.3.1 gives the result. $\square$

We now make the operad uA_∞ more explicit.

The underlying operad of the dg operad $\Omega\, u\mathcal{A}s^{\mathrm{i}}$ is the free operad $\mathcal{F}\left(s^{-1}\overline{\mathrm{q}u\mathcal{A}s^{\mathrm{i}}}\right) = \mathcal{F}\left(s^{-1}\left\{\mu_n^S\right\}\right)$, $n \geq 2$, $S \subset \underline{n}$ and $n = 1, S = \{1\}$, giving a free generating set for $\Omega\, u\mathcal{A}s^{\mathrm{i}}$. As a derivation of the composition structure, the differential $d = d_0 + 0 - d_2$ is completely determined by its action on the generators:

(1) 

where the last line means: for $(n,\, S) \neq (2,\, \{1\})$ and $(n,\, S) \neq (2,\, \{2\})$, we have

$$d(\overline{\mu}_n^S) = \sum_{\substack{p+q+r=n\\ p+1+r=m}} (-1)^{q(r+|S_1|)+|S_2||S_1'|+p+1} (\overline{\mu}_m^{S_1}; \underbrace{\mathrm{id},\, \ldots,\, \mathrm{id}}_{p-|S_1'|},\, \overline{\mu}_q^{S_2},\, \underbrace{\mathrm{id},\, \ldots,\, \mathrm{id}}_{r-|S_1''|}),$$

Remark. On the right-hand side of equation (1), the two-level trees now represent the compositions in the free operad.

We obtain the following description for a uA_∞-algebra structure.

6.1.9. **Proposition.** *A uA_∞-algebra structure on a dg module $(A,\, d_A)$ is given by a collection of maps, $\mu_1^{\{1\}}$, $\{\mu_n^S\}_{n\geq 2,\, S\subset \underline{n}}$ where each μ_n^S is a map $A^{\otimes(n-|S|)} \to A$ of degree $n + |S| - 2$ which together satisfy the following identities:*

$$\begin{cases} \partial(\mu_2^{\{1\}}) &=& \mu_2^{\emptyset} \circ (\mu_1^{\{1\}}, -) - \mathrm{id}_A \\ \partial(\mu_2^{\{2\}}) &=& \mu_2^{\emptyset} \circ (-, \mu_1^{\{1\}}) - \mathrm{id}_A \end{cases}$$

and for $(n,\, S) \neq (2,\, \{1\})$ and $(n,\, S) \neq (2,\, \{2\})$

$$\partial(\mu_n^S) = \sum_{\substack{p+q+r=n\\ p+1+r=m}} (-1)^{q(r+|S_1|)+|S_2||S_1'|+p+1} \mu_m^{S_1} \circ (\underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p-|S_1'|},\, \mu_q^{S_2},\, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{r-|S_1''|}).$$

Proof. Since uA_∞ is a quasi-free operad, a map $\mu_A : uA_\infty(A) \to A$ of degree 0 is determined by a collection of maps, $\mu_1^{\{1\}}$, $\{\mu_n^S\}_{n\geq 2,\, S\subset\underline{n}}$ where each μ_n^S is a map $A^{\otimes(n-|S|)} \to A$ of degree $n + |S| - 2$, defined by:

$$\mu_n^S(a_1 \otimes \cdots \otimes a_{n-|S|}) := \mu_A(\overline{\mu}_n^S \otimes a_1 \otimes \cdots \otimes a_{n-|S|}).$$

The fact that the map μ_A is a dg map gives the uA_∞ relations among the μ_n^S. $\square$

Remark. This notion of uA_∞-algebra corresponds to the notion of homotopy unit for an A_∞-algebra given in [FOOO09].

6.2. **Infinity-morphisms.** Following the classical case, we describe the *infinity-morphisms* of algebras over the Koszul resolution of a Koszul inhomogeneous quadratic operad. We give explicit formulae for infinity-morphism of uA_∞-algebras.

Unless we indicate otherwise, for the rest of this section, $\mathcal{P}$ will denote a Koszul inhomogeneous quadratic operad, $\mathcal{P}^{\mathrm{i}}$ its curved Koszul dual cooperad and $\mathcal{P}_\infty := \Omega\mathcal{P}^{\mathrm{i}}$ denotes the Koszul resolution of $\mathcal{P}$ (see Section 4).

Let A be a $\mathcal{P}_\infty$-algebra, and denote its structure map by $\mu_A \in \mathrm{Hom}_{\mathsf{dg\ op}}(\mathcal{P}_\infty, \mathrm{End}_A)$. Then by the bar-cobar adjunction 3.4.1, we have

$$\mathrm{Hom}_{\mathsf{dg\ operads}}(\Omega\mathcal{P}^{\text{i}}, \mathrm{End}_A) \cong \mathrm{Tw}(\mathcal{P}^{\text{i}}, \mathrm{End}_A).$$

By classical Hom-tensor duality, we have the bijection

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{S}\text{-}\mathsf{Mod}}(\mathcal{P}^{\text{i}}, \mathrm{End}_A) & \cong & \mathrm{Hom}_{\mathsf{dg\ mod}}(\mathcal{P}^{\text{i}}(A), A) \\ \mu_A & \longmapsto & d_{\mu_A}. \end{array}$$

We recall the classical lemma, that we can find for example in [LV].

6.2.1. **Lemma.** *A coderivation of $\mathcal{P}^i(A)$ is completely characterized by its corestriction to the cogenerators*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{mod.}}(\mathcal{P}^i(A), A) & \cong & \mathrm{Coder}(\mathcal{P}^i(A)) \\ d_{\mu_A} & \longmapsto & D^r_{\mu_A}. \end{array}$$

We call a *curved codifferential* any coderivation D of degree -1 which satisfies

$$D^2 = (\theta \circ \mathrm{id}_{\mathcal{P}^{\text{i}}(A)}) \circ \Delta_{\mathcal{P}^{\text{i}}(A)}.$$

We have the following extension of a classical result about codifferentials:

6.2.2. **Lemma.** *A $\mathcal{P}_\infty$-algebra structure on A is equivalent to a codifferential on $\mathcal{P}^i(A)$*

$$\begin{array}{ccc} \mathrm{Tw}(\mathcal{P}^i,\, \mathrm{End}_A) & \cong & \mathrm{curCodiff}(\mathcal{P}^i(A)) \\ \mu_A & \longmapsto & D_{\mu_A} := d_{\mathcal{P}^i(A)} + D^r_{\mu_A}. \end{array}$$

Proof. The predifferential $d_{\mathcal{P}^{\text{i}}}$ is a coderivation so the map $D_{\mu_A} := d_{\mathcal{P}^{\text{i}}(A)} + D^r_{\mu_A}$ is a coderivation. The construction here is the same as the construction in Section 5.2.3 with $D^r_{\mu_A} = d^r_{\mu_A} \circ_{\mathcal{P}} \mathrm{id}_A$, so $\mu_A \in \mathrm{Tw}(\mathcal{P}^{\text{i}},\, \mathrm{End}_A)$ implies $D^2_{\mu_A} = (\theta \circ \mathrm{id}_{\mathcal{P}^{\text{i}}(A)}) \circ \Delta_{\mathcal{P}^{\text{i}}(A)}$.

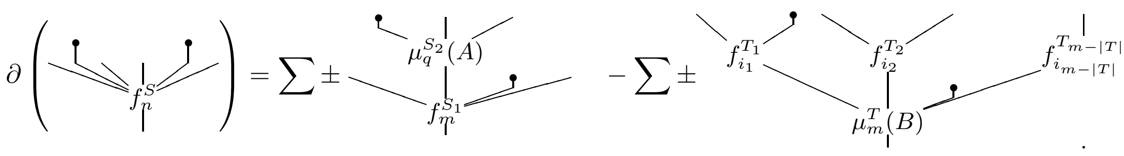
According to the proof of Lemma 5.2.4, we only have to remark that $D^r_{\partial(\mu_A)+\mu_A\star\mu_A-\Theta} = D^2_{\mu_A} - (\theta \circ id_{\mathcal{P}^{\text{i}}(A)}) \circ \Delta_{\mathcal{P}^{\text{i}}(A)} = 0$ implies $d_A \circ d_{\mu_A} + d_{\mu_A} \circ d_{\mathcal{P}^{\text{i}}(A)} + d_{\mu_A\star\mu_A} - d_\Theta = (D^r_{\partial(\mu_A)+\mu_A\star\mu_A-\Theta})^{|A} = 0$. Since $d_A \circ d_{\mu_A} + d_{\mu_A} \circ d_{\mathcal{P}^{\text{i}}(A)} + d_{\mu_A\star\mu_A} - d_\Theta$ is sent to $\partial(\mu_A) + \mu_A \star \mu_A - \Theta$ and 0 is sent to 0 by reversing the bijection in the Hom-tensor duality, we get the result. ☐

6.2.3. **Infinity-morphism of $\mathcal{P}_\infty$-algebras.** Let A and B be two $\mathcal{P}_\infty$-algebras, with structure maps μ_A and μ_B. A ∞-*morphism* $A \rightsquigarrow B$ *of $\mathcal{P}_\infty$-algebras* is a dg $\mathcal{P}^{\text{i}}$-coalgebra map

$$F : (\mathcal{P}^{\text{i}}(A),\, D_{\mu_A}) \to (\mathcal{P}^{\text{i}}(B),\, D_{\mu_B}).$$

This description of ∞-morphisms makes it clear that $\mathcal{P}_\infty$-algebras, ∞-morphisms, and composition given by composition of dg $\mathcal{P}^{\text{i}}$-coalgebra maps forms a category.

A $u\mathcal{A}s^{\text{i}}$-coalgebras map $F : u\mathcal{A}s^{\text{i}}(A) \to u\mathcal{A}s^{\text{i}}(B)$ is characterized by its corestriction to B, that is F is determined by a collection of maps $f^S_n : A^{\otimes(n-|S|)} \to B$. The fact that F commutes with the differentials is equivalent to a family of equations on the f^S_n. Pictorially, the collection of maps f^S_n satisfy:



6.2.4. **Proposition.** *Let A, B be two uA_∞-algebras, and let $\mu^S_n(A), \mu^S_n(B)$ be the respective structure maps. An ∞-morphism between A and B is a collection of maps*

$$\{f^S_n : A^{\otimes(n-|S|)} \to B\}_{n\geq 1,\, S\subset \underline{n}} \text{ of degree } n + |S| - 1,$$

satisfying: for $n = 1$, $d_A \circ f^\emptyset_1 = f^\emptyset_1 \circ d_A$, that is $f^\emptyset_1$ is a chain map, and for $n + |S| \geq 2$, $\partial(f^S_n) =$

$$\sum_{\substack{p+q+r=n \\ p+1+r=m}} (-1)^{p+q(r+|S_1|)+|S_2||S''_1|} f^{S_1}_m \circ (\underbrace{id_A,\, \ldots,\, id_A}_{p-|S'_1|},\, \mu^{S_2}_q(A),\, \underbrace{id_A,\, \ldots,\, id_A}_{r-|S''_1|})$$

$$+ \sum_{i_1-|T_1|+\cdots+i_{m-|T|}-|T_{m-|T|}|=n-|S|} -\epsilon(-1)^{(m+|T|-1)(n-m+|S|-|T|)}\mu_m^T(B)\circ\left(f_{i_1}^{T_1},\ \ldots,\ f_{i_{m-|T|}}^{T_{m-|T|}}\right),$$

where $\left\{\begin{array}{lcl} S_1' & \subset & \underline{p} \\ S_2 & \subset & \underline{q} \\ S_1'' & \subset & \{p+2,\ \ldots,\ p+1+r\}\end{array}\right.$ *such that* $S = S_1' \sqcup (S_2+p) \sqcup (S_1''+q-1)$ *and* $S_1 = S_1' \sqcup S_1''$*, where* $\left\{\begin{array}{lcl} T & \subset & \underline{m} \\ T_j & \subset & \underline{i_j}\end{array}\right.$ *such that* $T = R_0 \sqcup \ldots \sqcup R_{m-|T|}$ *and*

$$\begin{array}{rcl} S & = & R_0 \sqcup (T_1+|R_0|) \sqcup (R_1+i_1) \sqcup \ldots \sqcup \\ & & (T_{m-|T|}+|R_0|+\cdots+|R_{m-|T|-1}|+i_1+\cdots+i_{m-|T|-1}) \sqcup (R_{m-|T|}+i_1+\cdots+i_{m-|T|}) \end{array}$$

and where $\epsilon := |T|(n-m)+\sum_{j=1}^{m-|T|}\left[(i_j-1)(k-j+|T_1|+\cdots+|T_{j-1}|)+|R_j|(|T_1|+\cdots+|T_j|)\right]$.

Proof. An ∞-morphism $A \rightsquigarrow B$ is a $u\mathcal{A}s^{\text{!`}}$-coalgebra morphism $F : u\mathcal{A}s^{\text{!`}}(A) \to u\mathcal{A}s^{\text{!`}}(B)$. Such a morphism is completely determined by its image on the cogenerators of $u\mathcal{A}s^{\text{!`}}(B)$, that is by a map $f : u\mathcal{A}s^{\text{!`}}(A) \to B$ (of degree 0), or equivalently by a collection of maps $\{f_n^S : A^{\otimes(n-|S|)} \to B\}_{n\geq 1,\, S\subset \underline{n}}$ of degree $n+|S|-1$. The fact that F commutes with the predifferential is equivalent to the following commutative diagram

$$\begin{tikzcd} u\mathcal{A}s^{\text{!`}}(A) \arrow[r, "\Delta\circ id_A"] \arrow[d, "d_1+d_2"'] & u\mathcal{A}s^{\text{!`}}\circ u\mathcal{A}s^{\text{!`}}(A) \arrow[r, "id\circ f"] & u\mathcal{A}s^{\text{!`}}(B) \arrow[d, "d_B+d_2^{|B}"] \\ u\mathcal{A}s^{\text{!`}}(A) \arrow[rr, "f"'] & & B. \end{tikzcd}$$

Making this diagram explicit gives exactly the formulae of the Proposition. □

Example. For $n=1$ and $S=\{1\}$, the formula gives

$$\partial\left(f_1^{\{1\}}\right) = \begin{array}{c}\mu_1^{\{1\}}(A)\\ | \\ f_1^{\emptyset}\end{array} - \mu_1^{\{1\}}(B) \quad ,$$

that is, the element $f_1^{\{1\}}$ bounds the failure of $f_1^{\emptyset}$ to preserve the unit.

Remark. In [Lyu10], Lyubashenko proposes a definition for ∞-morphism between uA_∞-algebras as a resolution of bimodule. It would be interesting to compare his definition with our definition.

Before we end the section, we use the results above to give the following definition.

6.2.5. **Definition.** *A ∞-morphism of $\mathcal{P}_\infty$-algebras $F : A \rightsquigarrow B$ is a* quasi-isomorphism *if the chain map $f_1^{\emptyset} : A \to B$ induces an isomorphism in homology.*

6.3. **Rectification.** We now prove that for every uA_∞-algebra A there is a universal ∞-quasi-morphism I_A between A and a $u\mathcal{A}s$-algebra. This universal morphism takes the form of the unit of an adjunction. We make use of the bar and cobar constructions of algebras over Koszul operads (Sections 5.2.3, 5.2.5) for uA_∞-algebras and $u\mathcal{A}s$-algebras.

The twisting morphisms $\iota : u\mathcal{A}s^{\text{!`}} \to \Omega u\mathcal{A}s^{\text{!`}} = uA_\infty$ and $\kappa : u\mathcal{A}s^{\text{!`}} \to u\mathcal{A}s$ are defined in Section 3.4 and 4.3.

6.3.1. **Lemma.** *Let A be uA_∞-algebra. The morphism of dg $\mathbb{S}$-modules $A \rightarrowtail \Omega_\kappa \mathrm{B}_\iota A$ is a quasi-isomorphism.*

Proof. We endow $u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{!`}} \circ_\iota \Omega u\mathcal{A}s^{\text{!`}}$ with a filtration F_p given by

$$F_p(u\mathcal{A}s \circ u\mathcal{A}s^{\text{!`}} \circ \Omega u\mathcal{A}s^{\text{!`}}) = \bigoplus_{\omega+m\leq p} (u\mathcal{A}s \circ u\mathcal{A}s^{\text{!`}})^{(\omega)} \circ (\Omega u\mathcal{A}s^{\text{!`}})_m.$$

Moreover we endow $\Omega u\mathcal{A}s^{¡}$ with a filtration given by the homological degree, so that the morphism $\Omega u\mathcal{A}s^{¡} \rightarrowtail u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota \Omega u\mathcal{A}s^{¡}$ preserves the filtrations. Since the weight on $u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡}$ is non-negative and $\Omega u\mathcal{A}s^{¡}$ is non-negatively graded, the filtrations are bounded below. Moreover, the filtrations are exhaustive. Thus, we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

$$E^\bullet_{p,q} \Rightarrow \mathrm{H}_{p+q}(u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota \Omega u\mathcal{A}s^{¡}) \text{ and } {E'}^\bullet_{p,q} \Rightarrow \mathrm{H}_{p+q}(\Omega u\mathcal{A}s^{¡})$$

and an induced morphism between the spectral sequences. The differential on $E^0_{p,q}$ coincides with the differential on $\mathrm{q}u\mathcal{A}s \circ \mathrm{q}u\mathcal{A}s^{¡}$, so Proposition 6.1.6 shows that $E^1_{p,q} \cong {E'}^1_{p,q}$. It follows that $E^r_{p,q} \cong {E'}^r_{p,q}$ for all $r \geq 1$ and we get that $\Omega u\mathcal{A}s^{¡} \xrightarrow{\sim} u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota \Omega u\mathcal{A}s^{¡}$.

We have $\Omega_\kappa \mathrm{B}_\iota A \cong (u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota uA_\infty) \circ_{uA_\infty} A$. The short exact sequence

$$(u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota uA_\infty) \circ uA_\infty \circ A \to (u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota uA_\infty) \circ A \to (u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota uA_\infty) \circ_{uA_\infty} A$$

induces a long exact sequence in homology. Since we work over a field of characteristic 0, the ring $\mathbb{K}[\mathbb{S}_n]$ is semi-simple by Maschke's theorem, that is every $\mathbb{K}[\mathbb{S}_n]$-module is projective. So the Künneth formula implies that $\mathrm{H}_\bullet((u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota uA_\infty) \circ uA_\infty \circ A) \cong \mathrm{H}_\bullet(uA_\infty) \circ \mathrm{H}_\bullet(uA_\infty) \circ \mathrm{H}_\bullet(A)$ and $\mathrm{H}_\bullet((u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota uA_\infty) \circ A) \cong \mathrm{H}_\bullet(uA_\infty) \circ \mathrm{H}_\bullet(A)$. Finally, this gives that $\mathrm{H}_\bullet((u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡} \circ_\iota uA_\infty) \circ_{uA_\infty} A) \cong \mathrm{H}_\bullet(uA_\infty) \circ_{\mathrm{H}_\bullet(uA_\infty)} \mathrm{H}_\bullet(A) \cong \mathrm{H}_\bullet(A)$. ☐

6.3.2. **Theorem** (Universal rectification). *Let A be a uA_∞-algebra. There is a dg $u\mathcal{A}s$-algebra, $\Omega_\kappa \mathrm{B}_\iota A$ and an ∞-quasi-isomorphism $I_A : A \overset{\sim}{\rightsquigarrow} \Omega_\kappa \mathrm{B}_\iota A$ so that for any dg $u\mathcal{A}s$-algebra B and any ∞-morphism $F : A \rightsquigarrow B$, there is a unique dg $u\mathcal{A}s$-algebras map $\tilde{f} : \Omega_\kappa \mathrm{B}_\iota A \to B$ so that $F = \tilde{f} \circ I_A$, that is the following diagram commutes:*

$$\begin{tikzcd}
\Omega_\kappa \mathrm{B}_\iota A \arrow[dr, dashed, "\tilde{f}"] & \\
A \arrow[u, "I_A"] \arrow[r, "F"'] & B
\end{tikzcd}$$

Proof. The map I_A is defined by

$$i^S_n(a_1, \ldots, a_{n-|S|}) = \mu^S_n(a_1, \ldots, a_{n-|S|}) \in \mathrm{B}_\iota A \hookrightarrow \Omega_\kappa \mathrm{B}_\iota A.$$

By direct computation, this map is a ∞-morphism between the uA_∞-algebras A and $\Omega_\kappa \mathrm{B}_\iota A$. To see that this map is a quasi-isomorphism, observe that $i^\emptyset_1$ is equal to the inclusion map in Lemma 6.3.1. To define the map $\tilde{f}$, we note that the ∞-morphism of uA_∞-algebras F is determined by the collection of maps f^S_n, or by the collection of elements $\{f^S_n(a_1, \ldots, a_{n-|S|})\}$ in B. We define the module map $\mathrm{B}_\iota A \to B$ by

$$\mu^S_n(a_1, \ldots, a_{n-|S|}) \mapsto f^S_n(a_1, \ldots, a_{n-|S|}).$$

This map is a dg module map if and only if F is a ∞-morphism. Since the $u\mathcal{A}s$-algebra $\Omega_\kappa \mathrm{B}_\iota A$ is freely generated by $\{\mu^S_n(a_1, \ldots, a_{n-|S|})\}$, we define the map $\tilde{f}$ to be the lift of the above dg map to a $u\mathcal{A}s$-algebras map $\Omega_\kappa \mathrm{B}_\iota A \to B$. By construction we have $\tilde{f} \circ I_A = F$. ☐

Let us interpret the result above in terms of the categories of algebras. Since we have an operad map $uA_\infty \twoheadrightarrow u\mathcal{A}s$, we have an inclusion functor (one-to-one on objects and on morphisms) $u\mathcal{A}s\text{-}\mathsf{alg} \hookrightarrow uA_\infty\text{-}\mathsf{alg}$, which we denote by i. Denote by R the assignment that takes each uA_∞-algebra A to the $u\mathcal{A}s$-algebra $R(A) = \Omega_\kappa B_\iota(A)$. Because the arrow $A \xrightarrow{I_A} iR(A)$ is universal, R can be extended to morphisms so that it becomes a functor from uA_∞-alg $\to u\mathcal{A}s$-alg:

$$\begin{tikzcd}
R(A) \arrow[r, dotted, "R(F)"] & R(B) \\
A \arrow[u] \arrow[r, "F"] & B. \arrow[u]
\end{tikzcd}$$

We summarize in the following proposition.

6.3.3. **Proposition.** *The functor i, the object-assignment R, and the universal morphisms $A \xrightarrow{I_A} iR(A)$ determine the extension of R to a functor $R : uA_\infty$-alg $\to u\mathcal{A}s$-alg so that $I : \mathrm{id} \to iR$ is the unit of an adjunction:*

$$uA_\infty\text{-}\mathsf{alg} \underset{i}{\overset{R}{\rightleftarrows}} u\mathcal{A}s\text{-}\mathsf{alg}.$$

Proof. See Mac Lane [Mac98] chapter 4, theorem 1.

It is tempting to try to put a model category structure on the right-hand side so that this pair of functors becomes some kind of Quillen equivalence, as Lefevre-Hasegawa [LH03] did for A_∞-algebras and $\mathcal{A}s$-algebras. (Actually, A_∞-algebras are not quite a model category, see the referenced paper for more details). Instead we observe that each functor takes quasi-isomorphisms to quasi-isomorphisms, and so each functor induces a functor between the homotopy categories (localizations of each category by its quasi-isomorphisms). We claim these induced functors are an adjoint-equivalence of the homotopy categories.

6.4. **Transfer formulae.** In this section we provide formulae, based on labelled trees, for the pullback of a uA_∞-structure along a strong deformation retract.

For this entire section, suppose V, A are chain complexes, and

$$\begin{tikzcd} V \arrow[r, shift left, "i"] & A \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd},$$

is a strong deformation retract, i.e., p and i are chain maps, where $p \circ i = \mathrm{id}_V$ and $d_A h + h d_A = \mathrm{id}_A - i \circ p$. Moreover, suppose A is a uA_∞-algebra, with structure map μ_A.

6.4.1. **Definition.** *Let $n \geq 2, S \subset \underline{n}$, we define the set $\mathcal{T}_n^S$ be the set of planar, rooted trees, with n leaves, and a cork above each ith leaf if $i \in S$ which is labelled by either the word "connected" or "disconnected." We define $\mathcal{T}_1^\emptyset = \{\,|\,\}$ and $\mathcal{T}_1^{\{1\}} = \{\,\text{ꜛ}^{\text{connected}}\}$.*

6.4.2. **Definition.** *Let $T \in \mathcal{T}_n^S$, and let v be any internal vertex in T. We denote by $\mathrm{in}(v)$ the ordered (left-to-right) set of incoming edges to the vertex v. For each element $i \in \mathrm{in}(v)$, we define l_i and c_i as follows:*

1. *l_i is the total number of leaves without connected corks in the tree T whose (unique) path to the root passes through edge i*
2. *c_i is the total number of incoming edges to v without connected corks to the* right *of edge i.*

6.4.3. **Definition.** *For any $T \in \mathcal{T}_n^S$ and any internal vertex $v \in T$, we define*

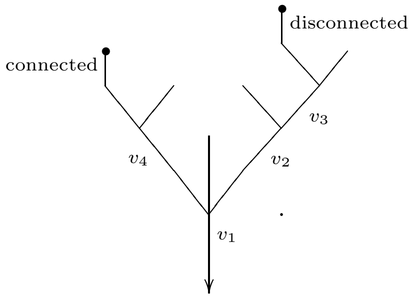
$$\epsilon(v) = \sum_{1 \leq i < j \leq |\mathrm{in}(v)|} (l_i + 1)\, l_j \quad + \sum_{\substack{i \in \mathrm{in}(v) \\ \text{with a connected} \\ \text{cork on it}}} c_i.$$

For any tree $T \in \mathcal{T}_n^S$, we set

$$\epsilon(T) = \sum_{\substack{\text{internal vertices} \\ v \in T}} \epsilon(v). \tag{2}$$

6.4.4. **Definition.** *Let $g_{structure} : \mathcal{T}_n^S \to \mathrm{Hom}(V^{\otimes(n-|S|)}, V)$ be the set map that takes an element $T \in \mathcal{T}_n^S$ and assigns to each vertex v the operation $\mu_{\mathrm{in}(v)}^{S(v)}(A)$ where $S(v)$ are the positions of the* connected *corks, the operation μ_1^1 to each disconnected cork, the homotopy h to each internal edge (that is not the outgoing edge of a connected cork), and the map i to each leaf without a cork above it, and the map p to the root of the tree. After this assignment, one composes the operations as indicated by internal edges to arrive at an operation $V^{\otimes(\mathrm{in}(v)-|S|)} \to V$. Let $g_{morphism} : \mathcal{T}_n^S \to \mathrm{Hom}(V^{\otimes(n-|S|)}, A)$ be the set map that takes an element $T \in \mathcal{T}_n^S$ and assigns to the tree the same element as $g_{structure}(T)$, but with the homotopy h assigned to the root, rather than the map p.*

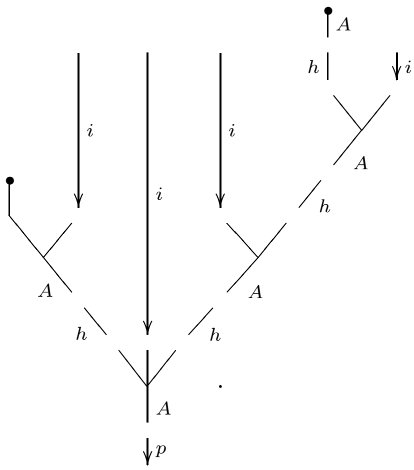
Examples. Let T be the element of $\mathcal{T}_5^{\{1,4\}}$ that looks like

$T =$ 

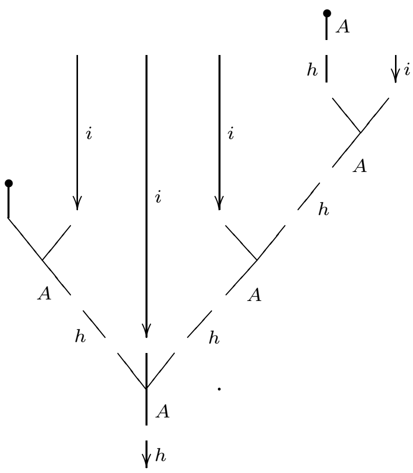
The sign $(-1)^{\epsilon(T)}$ for this tree is given by

$$\begin{aligned}
\epsilon(T) &= \epsilon(v_1) + \epsilon(v_2) + \epsilon(v_3) + \epsilon(v_4) \\
&= [(1+1)\cdot 1 + (1+1)\cdot 3 + (1+1)\cdot 3 + 0] + [(1+1)\cdot 2 + 0] + [(1+1)\cdot 1 + 0] + [(1+1)\cdot 1 + 1] \\
&= 14 + 4 + 2 + 3 \\
&= 23 \\
&\equiv 1 \bmod 2.
\end{aligned}$$

The operation assigned to the tree $T, g_{\text{structure}}(T)$, is given by the following composition of operations:

$g_{\text{structure}}(T) =$ 

while the morphism assigned to the tree $T, g_{\text{morphism}}(T)$ is given by:

$g_{\text{morphism}}(T) =$ 

6.4.5. **Proposition.** *The maps defined by*

$$\mu_n^S(V) := \sum_{T\in\mathcal{T}_n^S} (-1)^{\epsilon(T)} g_{structure}(T)\,. \tag{3}$$

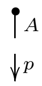
give V the structure of a uA_∞-algebra. Moreover, the maps defined by

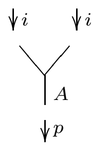
$$i_n^S := \sum_{T\in\mathcal{T}_n^S} (-1)^{\epsilon(T)} g_{morphism}(T)\,. \tag{4}$$

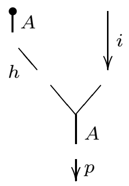
provide a ∞-quasi-isomorphism of uA_∞-algebras $I : V \stackrel{\sim}{\rightsquigarrow} A$.

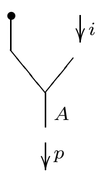
Proof. A combinatorial argument similar to the argument for transferring A_∞-structures [Mar06] will suffice. $\square$

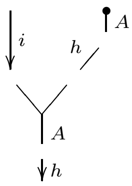
Examples. For small values of n, the transferred structure is given by

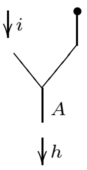
$\mu_1^{\{1\}}(V) := p\circ\mu_1^{\{1\}}(A) =$ 

$\mu_2^{\emptyset}(V) := p\circ\mu_2^{\emptyset}(A)\circ i^{\otimes 2} =$ 

$\mu_2^{\{1\}}(V) :=$ 

$-$ 

$i_2^{\{2\}}(V) :=$ 

$+$ 

For the reader familiar with transfer of A_∞-structures, restricting attention to the operations $\mu_n^{\emptyset}(V)$ recovers the familiar transfer formulae [Kad83, Mer99, KS06, Mar06, LV].

Remark. Though our signs differ from [Mar06], we use his ideas to develop a coherent sign convention for our transfer formulae. The reader should note that our function $\epsilon(v)$ differs from the $\theta(v)$ in [Mar06] even on the operations $\mu_n^{\emptyset}(V)$, in small ways, such as right-to-left orientation of trees instead of left-to-right. Instead we choose our signs to agree with [Sta63, LV] when restricted to the classical A_∞ operations.

6.5. **Comparing unital-(associative-infinity) and (unital-associative)-infinity.** In previous sections, we have developed the definition of the operad uA_∞ whose algebras are *homotopy unital* A_∞-algebras. There have been several definitions of homotopy unital A_∞-algebras [FOOO09, KS06, Lyu02], and these notions have been compared in [LM06]. There is also a definition of *strictly unital* A_∞-algebras [KS06, FOOO09]—we will refer to these as $\mathtt{su}A_\infty$-algebras throughout this section—they may be thought of as unital-(associative-infinity) algebras as opposed to our (unital-associative)-infinity algebras. We will compare uA_∞-algebras to $\mathtt{su}A_\infty$-algebras. This comparison includes Theorem 6.5.3, which states that every uA_∞-algebra has an equivalent unital-A_∞-structure on its homology. We demonstrate that this theorem is fairly general, and applies to many algebraic structures with units, including unital commutative associative algebras, unital Batalin-Vilkovisky algebras, and co-algebraic versions of these structures.

First we define $\mathtt{su}A_\infty$-algebras and their ∞-morphisms.

6.5.1. **Definition.** *An* $\mathtt{su}A_\infty$-algebra $(A, \{\mu_n\}_{n\geq 1}, e)$ *is an* A_∞*-algebra* $(A, \{\mu_n\}_{n\geq 1})$ *with* $e \in A$ *such that* $d_A(e) = 0$ *and* e *is a left and right unit for* μ_2*, and* e *annihilates* μ_n *for* $n \geq 3$ [KS06].

REMARKS. (1) There exists a dg-operad whose algebras are precisely $\mathtt{su}A_\infty$-algebras, and we denote it by $\mathtt{su}A_\infty$. Furthermore, the operad $\mathtt{su}A_\infty$ is the quotient of uA_∞ by the ideal generated by $\{\mu_n^S\}_{n\geq 2,\, |S|\geq 1}$. A quick computation yields that this map is a quasi-isomorphism.

(2) The operad $\mathtt{su}A_\infty$ is not cofibrant. If it were, the lifting property would imply that it is a retract of uA_∞ by the quotient map $uA_\infty \xrightarrow{\sim} \mathtt{su}A_\infty$, which a computation shows is impossible.

We now describe a diagram of categories of algebras. We will use the following notation

- $\mathcal{A}s$-alg: the category of associative algebras with algebra homomorphisms
- $u\mathcal{A}s$-alg: the category of unital associative algebras with algebra homomorphisms that preserve the unit
- ∞-A_∞-alg: the category of A_∞-algebras with ∞-morphisms
- ∞-uA_∞-alg: the category of uA_∞ algebras with ∞-morphisms
- $\mathtt{su}A_\infty$-alg: the category of $\mathtt{su}A_\infty$-algebras with the A_∞ ∞-morphisms for which f_1 preserves the unit and f_n annihilates it (for $n \geq 2$)

First, we have the following diagram of operads:

$$\begin{tikzcd}
u\mathcal{A}s & \mathsf{su}A_\infty \arrow[l, two heads, "\sim"'] & uA_\infty \arrow[l, two heads, "\sim"'] \\
\mathcal{A}s \arrow[u, tail] & & A_\infty \arrow[ll, two heads, "\sim"'] \arrow[u, tail]
\end{tikzcd}$$

On the categories of algebras, the diagram becomes:

$$\begin{tikzcd}
u\mathcal{A}s\text{-}\mathsf{alg} \arrow[r] \arrow[d] & \mathsf{su}A_\infty\text{-}\mathsf{alg} \arrow[r] & \infty\text{-}uA_\infty\text{-}\mathsf{alg} \arrow[d] \\
\mathcal{A}s\text{-}\mathsf{alg} \arrow[rr] & & \infty\text{-}A_\infty\text{-}\mathsf{alg}
\end{tikzcd}$$

We proved earlier (Section 6.3) that the first of the following composition of horizontal inclusions

$$\begin{cases} u\mathcal{A}s\text{-}\mathsf{alg} & \to & \infty\text{-}uA_\infty\text{-}\mathsf{alg}, \\ \mathcal{A}s\text{-}\mathsf{alg} & \to & \infty\text{-}A_\infty\text{-}\mathsf{alg} \end{cases}$$

has a left-adjoint, $\Omega_\iota B_\kappa$, which we called the universal rectification (it is known that the second has a similarly defined left-adjoint). Each of the inclusions,

$$\begin{cases} u\mathcal{A}s\text{-}\mathsf{alg} & \to & \mathcal{A}s\text{-}\mathsf{alg}, \\ \mathsf{su}\text{-}A_\infty\text{-}\mathsf{alg} & \to & A_\infty\text{-}\mathsf{alg} \end{cases}$$

has a left-adjoint as well, given by adjoining an element u and extending the product(s) to make u a strict unit (with appropriate annihilation conditions, in the case of $\mathtt{su}A_\infty$-algebras).

We now analyze the relationship between uA_∞ and $\mathtt{su}A_\infty$ via our transfer formulae.

6.5.2. **Theorem.** *Let* $V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h$ *, be a strong deformation retract, and* $\{\text{⫯}^A,\ \curlyvee_A\}$ *a strict* $u\mathcal{A}s$*-structure on* A*. Suppose further that* $h\left(\text{⫯}^A\right) = 0$*. Then the operations* $\mu_n^S(V)$ *given by the* *transfer formulae (see definition in Proposition 6.4.5) have the property that*

$$\mu_n^S(V) = 0$$

whenever $n \geq 2$ and $|S| \geq 1$. Furthermore, the uA_∞-morphism structure J on the chain map i has the property that whenever $|S| \geq 1$,

$$J_n^S = 0,$$

even when $n = 1$. In particular this means that the transferred uA_∞ structure is an $\mathtt{su}A_\infty$-algebra, and the uA_∞-∞ quasi-isomorphism is an $\mathtt{su}A_\infty$-∞ quasi-isomorphism.

Proof. For $n \geq 2, |S| \geq 1$, each summand in $\mu_n^S(V)$ contains as some part of the diagram (of compositions) the following composite:

$$\begin{array}{l} \dagger_A \\ |_h \end{array} = h(\dagger^A) = 0,$$

so each of those operations is itself 0. The same fact gives the result for J, along with the fact that

$$J_1^{\{1\}} = \begin{array}{l} \dagger_A \\ |_h \end{array} = 0.$$

The vanishing of these higher operations and morphisms implies that the transferred uA_∞ structure and morphism are strictly-unital, because the operad $\mathtt{su}A_\infty$ is the quotient of uA_∞ by precisely these operations. $\square$

Remark. We point out that since we are working over a field, and $d(\dagger^A) = 0$, it is always possible to choose a strong deformation retract between V and A so that $h\left(\dagger^A\right) = 0$ (provided, of course, V is equivalent to A).

The following corollary of Theorem 6.5.2 is an analogue of Theorem 5.4.2' in [FOOO09], which they prove in both the filtered and unfiltered setting.

6.5.3. **Corollary.** *Let A be a uA_∞-algebra. Then there exists a $u\mathcal{A}s$-algebra R, and an $\mathtt{su}A_\infty$-algebra structure on $H_\bullet(A)$ so that $A \xrightarrow{\sim} R$ and $H_\bullet(A) \xrightarrow{\sim} R$. That is, for an arbitrary uA_∞-algebra A, there is a minimal model for A which is an $\mathtt{su}A_\infty$-algebra.*

Proof. By Theorem 6.3.2, we have $I_A : A \overset{\sim}{\rightsquigarrow} \Omega_\kappa \mathrm{B}_\iota A = R(A)$. Note that in particular, $H_\bullet(A) \simeq_i H_\bullet(R(A))$. We will denote both by H.

Since there exist strong deformation retracts $H \underset{p}{\overset{i}{\rightleftarrows}} R(A) \circlearrowleft h$ where h annihilates the unit, transferring the $u\mathcal{A}s$ structure on $\Omega_\kappa \mathrm{B}_\iota A$ along any such strong deformation retract, by Theorem 6.5.2, gives an equivalent $\mathtt{su}A_\infty$-algebra structure on H. $\square$

In what follows, we prove an analogous theorem for a wide class of properads $\mathcal{P}$. First, we must say what we mean by a "unital version" of $\mathcal{P}$.

6.5.4. **Definition.** *Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be an inhomogeneous quadratic properad. We say an inhomogeneous quadratic properad $u\mathcal{P} = \mathcal{F}(\dagger \oplus V)/(R \oplus R')$ is a* unital version *of $\mathcal{P}$ if and only if*

- *the map of operads $\mathcal{P} \to u\mathcal{P}$ induced by the inclusion $V \to \dagger \oplus V$ is injective,*
- *the induced map $\mathrm{q}\mathcal{P} \to \mathrm{q}u\mathcal{P}$ together with the inclusion $\dagger \rightarrowtail \mathrm{q}u\mathcal{P}$ gives an isomorphism of operads $\dagger \oplus \mathrm{q}\mathcal{P} \simeq \mathrm{q}u\mathcal{P}$,*
- *the inhomogeneous quadratic relations associated to a single composition of the cork with an operation in $\mathcal{P}$ has only the leading quadratic term and a constant term.*

Remark. The name "unital version" for $u\mathcal{P}$ is not always appropriate. For example, if we take for $\mathcal{P}$ the operad $\mathcal{L}ie$, then the operad $c\mathcal{L}ie$, which governs Lie algebras with a designated central element, is a unital version of $\mathcal{L}ie$ as in the above (where the constant term is taken to be zero), though of course a central element is far from what we typically think of as a unit.

Suppose $u\mathcal{P}$ is a unital version of $\mathcal{P}$, and that both are inhomogeneous Koszul properads. Then $\Omega u\mathcal{P}^{¡} =: u\mathcal{P}_\infty \xrightarrow{\sim} u\mathcal{P}$, and by Proposition 6.1.4, the underlying coproperad of $u\mathcal{P}^{¡}$ is isomorphic to $\mathbf{¡}^{¡} * \mathrm{q}\mathcal{P}^{¡}$. This observation allows us to define "strictly-unital $\mathcal{P}_\infty$-algebras," or $\mathtt{su}\mathcal{P}_\infty$-algebras, as we defined $\mathtt{su}A_\infty$: we can identify the "unital homotopies" as those made of a (co)operation $\mu_\alpha \in \mathrm{q}\mathcal{P}^{¡}$ with some configuration S of corks above the leaves.

6.5.5. **Definition.** *Suppose $u\mathcal{P}$ is a unital version of $\mathcal{P}$, and that both are inhomogeneous Koszul properads. We define the properad $\mathtt{su}\mathcal{P}_\infty$ as the quotient of $u\mathcal{P}_\infty$ by the (differential) properadic ideal generated by the operations*

$$\left\{\mu_\alpha^S : \quad \text{for } \mu_\alpha \in \mathrm{q}\mathcal{P}^{i} \text{ and } S \neq \emptyset\right\}.$$

That is, we quotient by all the "unital relations" and "unital homotopies."

Remark. Though it looks like we only quotient by unital homotopies in the above, taking the *differential* ideal generated by the unital homotopies means we also quotient by the image under $d_{u\mathcal{P}_\infty}$ of the unital homotopies with weight 2, which are precisely the unital relations.

If $u\mathcal{P}$ is a unital version of $\mathcal{P}$, and both are inhomogeneous Koszul properads, the quotient map $\mathtt{su}\mathcal{P}_\infty \twoheadrightarrow u\mathcal{P}$ is a quasi-isomorphism. In general, however, the operad $\mathtt{su}\mathcal{P}_\infty$ is not cofibrant. Even so, we have the following transfer theorem for $\mathtt{su}\mathcal{P}_\infty$-algebras.

6.5.6. **Theorem.** *Let $u\mathcal{P}$ be a unital version of $\mathcal{P}$, and suppose both are inhomogeneous Koszul. Then given any $u\mathcal{P}$-algebra A and a strong deformation retract $V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h$, where the homotopy h satisfies $h(\mathbf{¡}^A) = 0$, the transferred $(u\mathcal{P})_\infty$-algebra is an $\mathtt{su}\mathcal{P}_\infty$-algebra structure, and the $u\mathcal{P}_\infty$ ∞-morphism structure on J is an $\mathtt{su}\mathcal{P}_\infty$ ∞-morphism.*

6.5.7. **Corollary.** *Suppose we have properads $\mathcal{P}, u\mathcal{P}$ as in Theorem 6.5.6, and suppose A is a $u\mathcal{P}$-algebra. Then there is an $\mathtt{su}\mathcal{P}_\infty$-algebra structure on the homology of A and an $\mathtt{su}\mathcal{P}_\infty$ ∞-quasi isomorphism $H \overset{\sim}{\rightsquigarrow} A$.*

Proof. It is a corollary of the proof for $u\mathcal{A}s$, given the universal rectification and transfer formulae for arbitrary Koszul inhomogeneous quadratic properads $u\mathcal{P}$ (which are not made explicit in this paper). $\square$

6.5.8. **Corollary.** *In the following list of pairs, $(\mathcal{P}, u\mathcal{P})$, $u\mathcal{P}$ is a unital model for $\mathcal{P}$ and both are inhomogeneous Koszul. In particular, each $u\mathcal{P}$-algebra structure may be transferred to an equivalent $\mathtt{su}\mathcal{P}_\infty$ structure on homology in the above sense.*

1. *$(\mathcal{C}om, u\mathcal{C}om)$, where $u\mathcal{C}om$ is the operad governing unital commutative associative algebras,*
2. *$(\mathcal{L}ie, c\mathcal{L}ie)$, where $c\mathcal{L}ie$ is the operad governing Lie algebras with a designated central element,*
3. *$(\mathcal{G}erst, u\mathcal{G}erst)$, where $u\mathcal{G}erst$ is the operad governing unital Gerstenhaber algebras, ie, Gerstenhaber algebras with a unit for the commutative associative product which is annihilated by the bracket,*
4. *$(\mathcal{BV}, u\mathcal{BV})$, where $u\mathcal{BV}$ is the operad governing unital $\mathcal{BV}$-algebras, ie, BV algebras with a unit for the commutative associative product which is annihilated by the bracket and the delta operator (see* [GCTV09] *for a treatment of $\mathcal{BV}$ as an inhomogeneous Koszul operad).*

Remark.

1. Though we have spoken only about units, counits may be treated similarly.
2. Treating $uc\mathcal{F}rob$, the properad governing Frobenius algebras with unit and counit, would be interesting to the authors.

6.6. **Cohomology theory for unital associative algebra.** In this section, we define the André-Quillen cohomology theory for unital associative algebras following the general definition of [Mil08]. We prove that the cohomology can be written as an Ext-functor and we compare this definition to the Hochschild cohomology theory.

6.6.1. **André-Quillen cohomology theory.** We consider now the operad $\mathcal{P} = u\mathcal{A}s$ and the curved cooperad $\mathcal{C} = u\mathcal{A}s^{¡} = (\mathrm{q}u\mathcal{A}s^{¡},\, 0,\, \theta)$. The Koszul morphism between $u\mathcal{A}s$ and $u\mathcal{A}s^{¡}$ is given by

$$\kappa : u\mathcal{A}s^{¡} \twoheadrightarrow \text{¦} \oplus \curlyvee \rightarrowtail u\mathcal{A}s.$$

Let A be a $u\mathcal{A}s$-algebra. Following Sections 1 and 2 of [Mil08], we use the cofibrant resolution

$$\Omega_\kappa \mathrm{B}_\kappa A = u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{¡}(A) \xrightarrow{\sim} A$$

of Section 5 to compute the André-Quillen cohomology of A thanks to the cotangent complex

$$A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{¡}(A) \cong \underbrace{A \;\; u\mathcal{A}s^{¡}(A) \;\; A} \cong A \otimes u\mathcal{A}s^{¡}(A) \otimes A.$$

We denote an element in $A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{¡}(A)$ by $a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c$, where a, b_t and c are in A and where $\overline{\mu}_n^S$ is in $u\mathcal{A}s^{¡}(n-|S|)$. Following the end of Section 2 of [Mil08], we compute the differential on $A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{¡}(A)$, which is given by

$$d_\varphi := d_{A\otimes^{u\mathcal{A}s} u\mathcal{A}s^{¡}(A)} - \delta_\varphi^l + \delta_\varphi^r.$$

The differential $d_{A\otimes^{u\mathcal{A}s} u\mathcal{A}s^{¡}(A)}$ depends only on d_A (since $d_{u\mathcal{A}s} = 0$, $d_{u\mathcal{A}s^{¡}} = 0$), the map $\varphi : u\mathcal{A}s^{¡}(A) \twoheadrightarrow A$ is the projection and the terms δ_φ^l and δ_φ^r are given by the following proposition.

6.6.2. **Proposition.** *We have*
$\delta_\varphi^l(a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c) :=$

$$\epsilon_1 a \cdot b_1 \otimes (\overline{\mu}_{n-1}^{S-1} \otimes b_2 \cdots b_{n-|S|}) \otimes c + (-1)^n \epsilon_{n-|S|} a \otimes (\overline{\mu}_{n-1}^{S} \otimes b_1 \cdots b_{n-|S|-1}) \otimes b_{n-|S|} \cdot c,$$

where $\epsilon_i := \begin{cases} (-1)^{|a|+|b_i|(n-2+|S|+|b_1|+\cdots+|b_{i-1}|)} & \text{if } i \notin S, \\ 0 & \text{otherwise,} \end{cases}$ *and*
$\delta_\varphi^r(a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c) := (\delta_{\text{¦}} + \delta_\gamma)(a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c) =$

$$- \sum_{S=S_1 \sqcup \{u\} \sqcup S_1'} (-1)^{|a|+n+|S_1|} a \otimes (\overline{\mu}_n^{S\setminus u} \otimes b_1 \cdots 1_A \cdots b_{n-|S|}) \otimes c$$

$$- \sum_{\{t,\, t+1\} \sqcup S = S_2 \sqcup \{t,\, t+1\} \sqcup S_2'} (-1)^{|a|+t+|S|} a \otimes (\overline{\mu}_{n-1}^{S_2 \sqcup \{S_2'-1\}} \otimes b_1 \cdots b_t \cdot b_{t+1} \cdots b_{n-|S|}) \otimes c,$$

where $\max S_1 < u < \min S_1'$ *and* $\max S_2 < t < t+1 < \min S_2'$ *and* $\delta_{\text{¦}}$ *holds for the first sum and* δ_γ *for the second. Moreover,* $d_\varphi(\text{¦}) = 0$.

PROOF. The differential on the cotangent complex is given following the end of Sections 2 of [Mil08]. We make the computations explicit thanks to the infinitesimal decomposition map of $u\mathcal{A}s^{¡}$, described in Corollary 6.1.5. ☐

6.6.3. **Proposition.** *The André-Quillen cohomology groups of a $u\mathcal{A}s$-algebra A with coefficients in a unital A-bimodule M are given by*

$$\mathrm{H}^\bullet_{u\mathcal{A}s}(A,\, M) := \mathrm{H}_\bullet(\mathrm{Hom}_{\mathsf{A\text{-}bimod.}}(A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{i}(A),\, M),\, \partial),$$

where $\partial(f) := d_M \circ f - (-1)^{|f|} f \circ d_\varphi$ *and* $\mathsf{A\text{-}bimod.}$ *is the category of unital A-bimodules.*

6.6.4. **Ext-functor and comparison with the Hochschild cohomology theory.** To a unital associative algebra, we can associate two abelian groups: the Hochschild cohomology groups of A (as defined in [Hoc45], or [Lod98], chap. 1, for a modern reference), that is, the André-Quillen cohomology groups of the associative algebra A (forgetting the unit), or the André-Quillen cohomology groups of A seen as a unital associative algebra (previous section). We show that the cohomology groups coincide.

6.6.5. **Theorem.** *Let A be a $u\mathcal{A}s$-algebra and let M be a unital A-bimodule. We have*

$$\mathrm{H}^\bullet_{u\mathcal{A}s}(A,\, M) \cong \mathrm{Ext}^\bullet_{A\otimes^{u\mathcal{A}s}\mathbb{K}}(\Omega_{u\mathcal{A}s}(A),\, M),$$

where $\Omega_{u\mathcal{A}s}(A)$ is the unital A-bimodule of Kähler differential forms (see [Mil08] *for more details).*

Proof. Similarly to the case of Hochschild cohomology theory, we define the map h on $A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A$ by

$$h(a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c) = -(-1)^{|a|(n+|S|)} 1 \otimes (\overline{\mu}_{n+1}^{S+1} \otimes ab_1 \cdots b_{n-|S|}) \otimes c.$$

It satisfies $dh + hd = id$ on $A \otimes \overline{u\mathcal{A}s^{\text{i}}}(A) \otimes A$. Thus the chain complex

$$A \otimes \overline{u\mathcal{A}s^{\text{i}}}(A) \otimes A \xrightarrow{d_\varphi} A \otimes A \otimes A \twoheadrightarrow \Omega_{u\mathcal{A}s}(A) \to 0$$

is acyclic since we derive the left-adjoint functor of Kähler differential forms to obtain the cotangent complex, and the cohomology is an Ext-functor. $\square$

We use this theorem to compare this cohomology theory to the Hochschild cohomology theory.

6.6.6. **Proposition.** *There is a quasi-isomorphism of unital A-bimodules*

$$A \otimes^{u\mathcal{A}s} \mathcal{A}s^{i}(A) \cong A \otimes \mathcal{A}s^{i}(A) \otimes A \xrightarrow{\sim} A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{i}(A) \cong A \otimes u\mathcal{A}s^{i}(A) \otimes A.$$

Proof. First, we endowed $A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A$ with a filtration given by the number of corks, denoted by

$$F_p(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A) := \bigoplus_{S \subseteq \underline{n},\, |S| \leq p} A \otimes (u\mathcal{A}s^{\text{i}}(n - |S|) \otimes_{\mathbb{S}_{n-|S|}} A^{\otimes(n-|S|)}) \otimes A.$$

We have $d_{A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{i}}(A)} : F_p \to F_p$, $\delta_\varphi^l : F_p \to F_p$, $\delta_{\text{┯}} : F_p \to F_{p-1}$ and $\delta_\gamma : F_p \to F_p$. Thus the filtration is a filtration of chain complexes. It is bounded below and exhaustive so we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) and we obtain a spectral sequence $E_{p,q}^\bullet$ such that

$$E_{p,q}^\bullet \Rightarrow \mathrm{H}_{p+q}(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A).$$

The differential d^0 on $E_{p,q}^0 := F_p(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A)_{p+q}/F_{p-1}(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A)_{p+q}$ is given by $d^0 = d_{A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{i}}(A)} - \delta_\varphi^l + \delta_\gamma$. There is an inclusion of chain complexes

$$i : A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A \rightarrowtail \oplus_{p,\,q} E_{p,q}^0 \cong A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A,$$

where the last isomorphism is only of vector spaces. The projection $p : \oplus_{p,\,q} E_{p,q}^0 \cong A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A \oplus C_{\geq 1} \twoheadrightarrow A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A$, where $C_{\geq 1}$ is given by elements with at least one cork, is a chain complexes map. We define the map h by

$$h(a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c) := -(-1)^{\min S} a \otimes (\overline{\mu}_{n+1}^{S+1} \otimes b_1 \cdots b_{(\min S)-1} 1_A b_{\min S} \cdots b_{n-|S|}) \otimes c.$$

With these definitions, we have $p \circ i = id_{A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A}$ and $id_{\oplus_{p,\,q} E_{p,q}^0} - i \circ p = dh + hd$. Hence, we have a deformation retract

$$\begin{tikzcd} A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A \arrow[r, shift left, "i"] & \oplus_{p,\,q} E_{p,q}^0 \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}$$

and the inclusion i is a quasi-isomorphism. It follows that $E_{p,q}^1 = 0$ when $p \neq 0$ and the spectral sequence collapses. Considering the filtration $F_p'(A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A) = A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A$ for all $p \geq 0$ (bounded below and exhaustive), the inclusion induces a map of spectral sequences which is a quasi-isomorphism on the E^1-pages and higher. Since ${E'}_{p,q}^\bullet$ converges to $\mathrm{H}_{p+q}(A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A)$ and $E_{p,q}^\bullet$ converges to $\mathrm{H}_{p+q}(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A)$, we get the proposition. $\square$

6.6.7. **Corollary.** *Let A be a unital associative algebra. For $\bullet \geq 1$, we have*

$$\mathrm{H}_{u\mathcal{A}s}^\bullet(A,\, M) \cong \mathrm{HH}^{\bullet+1}(A,\, M).$$

Proof. The cohomology of $u\mathcal{A}s$-algebras is given by the Ext-functor $\mathrm{Ext}_{A \otimes^{u\mathcal{A}s} \mathbb{K}}^\bullet(\Omega_{u\mathcal{A}s}(A),\, M)$ (Theorem 6.6.5) and we have the projective resolution $A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A \xrightarrow{\sim} \Omega_{u\mathcal{A}s}(A)$. By Proposition 6.6.6, the projective (quasi-free) A-bimodule $A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A$ is also a projective resolution of $\Omega_{u\mathcal{A}s}(A)$ and computes the Hochschild cohomology (see the definition 1.1.3 in [Lod98]). $\square$

Acknowledgments

The authors thank Bruno Vallette for his time, his help and all his comments. The first author is grateful to Gabriel Drummond-Cole for his tutelage in Koszul Duality while this project was in its early stages, and for many helpful discussions throughout. He would also like to thank John Terilla for his insights and encouragement. The second author is grateful to Éric Hoffbeck for his comments on the preliminary version of the paper and to Benoit Fresse who stressed a mistake in the proof of Proposition 5.2.8. He is also grateful to the Max-Planck-Institut für Mathematik in Bonn for the excellent working conditions he found there between September 2009 and January 2010.

References

[Abr96] L. Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory Ramifications*, 5, 1996.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin, 1973.

[FOOO09] K. Fukaya, Y. Oh, H. Ohta, and K. Ono. In *Lagrangian Intersection and Floer Theory: Anomaly and Obstruction*, volume 46.1-46.2 of *AMS/IP studies in advanced mathematics.* Amer. Math. Soc. and International Press, Providence, RI; Somerville, MA, 2009.

[Fre09] B. Fresse. *Modules over operads and functors.* Lecture Notes in Mathematics, No. 1967. Springer-Verlag, 2009.

[Fuk02] K. Fukaya. Floer homology and mirror symmetry. II. *Adv. Stud. in Pure Math*, 34:31–127, 2002.

[GCTV09] I. Gàlvez-Carrillo, A. Tonks, and B. Vallette. Homotopy Batalin-Vilkovisky algebras. 2009. arXiv:0907.2246.

[GJ94] E. Getzler and J. D. S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces. preprint http://arxiv.org/hep-th/9403055, 1994.

[GK94] V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76, 1994.

[Hoc45] G. Hochschild. On the cohomology groups of an associative algebra. *Ann. of Math. (2)*, 46:58–67, 1945.

[Kad83] T. Kadeishvili. The algebraic structure in the homology of an $A(\infty)$-algebra. *Soobshch. Akad. Nauk Gruzin. SSR*, no. 2(108):249–252, 1983.

[Koc04] J. Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 2004.

[KS06] M. Kontsevich and Y. Soibelman. Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. i. 2006. preprint http://arxiv.org/abs/math/0606241.

[LH03] K. Lefevre-Hasegawa. Sur les A-infini catégories, 2003. arXiv:0310337.

[LM06] V. Lyubashenko and O. Manzyuk. Unital A infinity categories. *Problems of topology and related questions*, (3):235–268, 2006.

[Lod98] J.-L. Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, second edition, 1998.

[Lod01] J.-L. Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001.

[LV] J.-L. Loday and B. Vallette. *Algebraic operads.* http://math.unice.fr/∼brunov/Operades.html.

[Lyu02] V. Lyubashenko. Category of A_∞ categories. 2002. preprint http://arxiv.org/abs/math/0210047/.

[Lyu10] Homotopy unital A_∞-algebras. *Journal of Algebra*, In Press, Corrected Proof, 2010.

[Mac98] S. Maclane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, Berlin, Heidelberg, 2nd ed edition, 1998.

[Mar01] M. Markl. Ideal perturbation lemma. *Comm. Algebra*, 29, 2001.

[Mar06] M. Markl. Transfering A_∞ (strongly homotopy associative) structures. In *Proceedings of the 25th winter school "Geometry and Physics"*, pages 139–151. Circ. Mat. Palermo, Palermo, Italy, 2006.

[May72] J. P. May. *The geometry of iterated loop spaces.* Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[Mer99] S. Merkulov. Strongly homotopy algebras of a Kähler manifold. *Internat. Math. Res. Notices*, (no. 3):153–164, 1999.

[Mil08] J. Millès. André-Quillen cohomology of algebras over an operad. 2008. preprint http://arxiv.org/abs/0806.4405.

[MSS02] M. Markl, S. Shnider, and J. Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2002.

[MV07] S. Merkulov and B. Vallette. Deformation theory of representation of prop(erad)s ii. 2007. To appear in J. Reine Angew. Math. (Crelle), arXiv:0707.0889.

[MV09] S. Merkulov and B. Vallette. Deformation theory of representation of prop(erad)s i. *J. Reine Angew. Math.*, 634:51–106, 2009.

[Nic08] P. Nicolás. The bar derived category of a curved dg algebra. *J. Pure Appl. Algebra*, 212(12), 2008.

[Pos93] L. E. Positsel′skiĭ. Nonhomogeneous quadratic duality and curvature. *Funktsional. Anal. i Prilozhen.*, 27:57–66, 96, 1993.

[Pos09] L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence, 2009. http://arxiv.org/abs/0905.2621.

[PP05] A. Polishchuk and L. Positselski. *Quadratic algebras*, volume 37 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2005.

[Pri70] S. B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970.

[Sta63] J. D. Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108, 1963.

[Val07] B. Vallette. A Koszul duality for PROPs. *Trans. Amer. Math. Soc.*, 359(10):4865–4943 (electronic), 2007.

[Val08] B. Vallette. Manin products, Koszul duality, Loday algebras and Deligne conjecture. *J. Reine Angew. Math.*, 620:105–164, 2008.

[van03] P. van der Laan. Coloured Koszul duality and strongly homotopy operads. 2003. arXiv:math/0312147.

[Wei94] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Wil07] S. O. Wilson. Free frobenius algebra on the differential forms of a manifold, 2007. preprint http://arxiv.org/pdf/0710.3550v2.

Joan Millès, Laboratoire J. A. Dieudonné, Université de Nice - Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
E-mail : `joan.milles@math.unice.fr`
URL : `http://math.unice.fr/~jmilles`

Joseph Hirsh, CUNY Graduate Center, 365 Fifth Avenue, New York NY 10016
E-mail : `jhirsh1@GC.cuny.edu`